Notes on Dynamical Systems

1 Preliminaries

1.1 Metric Spaces

Definition 1.1 (Metrics)**.** *Let X be a set. A map $d : X \times X \to [0, \infty)$ is called a metric on X if*

1. $d(x, y) = d(y, x)$,
2. $d(x, y) = 0$ *if and only if* $x = y$,
3. $d(x, y) + d(y, z) \geq d(x, z)$.

(X, d) is called a metric space, where d is a metric and X is a topological space with topology induced by d. We also denote it by X when the metric is indicated.

Definition 1.2 (Complete metric space)**.** *Let (X, d) be a metric space. A sequence $\{x_k\}_{k=1,2,...}$ in X is called a Cauchy sequence if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n, m > N$ we have $d(x_n, x_m) < \epsilon$.*

We call a metric space X is complete, if every Cauchy sequence in X converges in M.

Example 1.3 (The space of continuous maps)**.** *Let X be a compact topological space, Y be a metric space with a metric d, and $C(X, Y)$ be the set of all continuous maps from X to Y.*

Define a metric dist_0 on $C(X, Y)$ by

$$\mathrm{dist}_0(f, g) = \min\{1, \sup_{x \in X} \max\{d(f(x), g(x))\}\}$$

for $f, g \in C(X, Y)$.

Note that if Y is a complete metric space, then $C(X, Y)$ is complete as well, because its topology is as same as that induced by the uniform metric when X is compact.

1.2 Riemannian manifolds

In this subsection, we give a brief introduction to Riemannian manifolds.

Definition 1.4 (C^k Riemannian manifolds)**.** *A C^k Riemannian metric is a family of positive definite symmetric bilinear form $\{\langle\ ,\ \rangle_p\}_{p \in M}$ defined on the tangent space T_pM with the following property: For any C^k vector fields X and Y, the map $p \to \langle X_p, Y_p \rangle_p$ is C^k.*

A C^k Riemannian manifold is a C^k manifold with a C^k Riemannian metric.

When $k = \infty$, we call it a smooth Riemannian manifold.

In this note, we mainly focus on the smooth case.

Let $x = (x^1, ..., x^d)$ be local coordinates. The metric can be represented by a positive definite, symmetric matrix

$$(g_{ij}(x))_{i,j=1,...,d},$$

where $g_{ij}(x)$ is smooth. And we denote the inverse of the metrix by $(g^{ij}(x))_{i,j=1,...,d}$.

For every $v \in T_pM$, we define $\|v\| = (\langle v, v \rangle_p)^{1/2}$. Let $[a, b]$ be a closed interval in $\mathbb{R}$, and $\gamma : [a, b] \to M$ be a smooth curve. We define the energy of λ by

$$E(\gamma) = \frac{1}{2} \int_a^b \|\frac{\mathrm{d}\gamma}{\mathrm{d}t}\mathrm{d}t\|^2.$$

We now rewrite it in local coordinates:

$$E(\gamma) = \frac{1}{2} \int_a^b g_{ij}(x(\gamma(t)))\dot{x}^i(t)\dot{x}^j(t),$$

where the local coordinates of $\gamma(t)$ is $x(\gamma(t)) = (x^1(\gamma(t)), ..., x^d(\gamma(t))$, and $\dot{x}^i(t) = \frac{d}{dt}(x^i(\gamma(t)))$.

Then the Euler-Lagrange equations for the energy E are

$$\ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0, \; i = 1, ..., d$$

with the Christoffel symbols Γ^i_{jk} defined by

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}(g_{jl,k} + g_{kl,j} - g_{jk,l}),$$

where $g_{jl,k} = \frac{\partial}{\partial x^k} g_{jl}$.

Definition 1.5 (Geodesic)**.** *For a Riemannian manifold M, a differentiable curve $\gamma : [a, b] \to M$ is called a geodesic if it satisfies the Euler-Lagrange equations for the energy $E(\gamma)$.*

It is well-known that the geodesics are the shortest curves between two points which are sufficiently close and there always exist geodesics on compact manifolds. Moreover, for a compact Riemannian manifold M, any $p \in M$, $v \in T_pM$, there is a unique geodesic

$$c_v : (-\infty, +\infty) \to M$$

with $c(0) = p, \dot{c}(0) = v$; and c_v continuously depends on p and v.

Definition 1.6 (Exponential map)**.** *Let M be a compact Riemannian manifold, $p \in M$. We define the exponential map*

$$\exp_p : T_pM \to M, \; \textit{by } v \to c_v(1).$$

It is necessary to point out when M is not compact, the exponential map $\exp_p$ may not defined on the whole of T_pM. Fortunately, in this note, we only focus on the compact case, so $\exp_p$ is defined on the entire T_pM for every $p \in M$ by the Hopf-Rinow theorem[24, p.34].

Example 1.7 (Riemannian manifold as a metric space)**.** *Let M be a connected compact C^k Riemannian manifold. For any $x, y \in M$, there is a C^1 path connecting x and y, that is, a C^1 map $\gamma : [0, 1] \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$. We can define the length of γ by*

$$L(\gamma) = \int_0^1 \| \frac{\mathrm{d}\gamma}{\mathrm{d}t} \mathrm{d}t \|.$$

Now we have

$$d(x, y) = \inf\{L(\gamma) : \gamma \textit{ is a } C^1 \textit{ path connecting } x \textit{ and } y\},$$

which makes M a metric space.

Moreover, M is a complete metric space by the Hopf-Rinow theorem.

Example 1.8 (The space of C^1 maps)**.** *Let M,N be C^1 Riemannian manifolds, $C^1(M, N)$ be the set of all C^1 maps from M to N.*

Recall that the Riemannian metric on M induces a metric on the tangent bundle TM naturally.

Define a metric dist_1 on $C^1(M, N)$ as follow:

$$\mathrm{dist}_1(f, g) = \mathrm{dist}_0(df, dg)$$

for $f, g \in C^1(M, N)$.

1.3 Topological Dynamics

In this subsection, X denotes a locally compact separable metric space, and f is always continuous.

Definition 1.9 (Topological dynamics)**.** *A topological space X with a continuous map $f : X \to X$ is called a discrete-time dynamical system.*

In this note, every dynamical system means the discrete-time dynamical system.

Definition 1.10 (Periodic points). *A point $x \in X$ is called periodic point of $f : X \to X$ with period $n \in \mathbb{N}$, if $f^n(x) = x$. The set of all periodic points of $f : X \to X$ is denoted by* $\mathrm{Per}(f)$. *The smallest positive $n \in \mathbb{N}$, such that $f^n(x) = x$ is called minimal period of x.*

Definition 1.11 (Non-wondering points). *A point $x \in X$ is called a non-wondering point of $f : X \to X$, if for any neighborhood U of x, there exists $n \in \mathbb{N}$ such that $f^n(U) \cap U$ is a non-empty set. The set of all non-wondering points of $f : X \to X$ is denoted by* $\mathrm{NW}(f)$.

Definition 1.12 (Topological transitivity). *A topological dynamical system $f : X \to X$ is called topologically transitive if for any two non-empty open sets U and V, $\exists N \in \mathbb{Z}$ such that $f^N(U) \cap V \neq \emptyset$.*

Definition 1.13 (Topological mixing). *A topological dynamical system $f : X \to X$ is called topologically mixing if for any two non-empty open sets U and V, $\exists N > 0$ such that $f^n(U) \cap V \neq \emptyset$ for all $n > N$.*

From the definitions above, it is obvious that if a topological dynamical system is topologically mixing then it is topologically transitive. Usually, there are topologically transitive maps which are not topologically mixing, such as an irrational rotation of the circle. However, for Anosov diffeomorphisms, topological mixing is equivalent to topological transitivity, which would be shown in Section 4.2.

Definition 1.14 (Topological conjugacy). *A map $f : M \to M$ is called topologically conjugate to a map $g : N \to N$ if there exist a homeomorphism $h : M \to N$ such that $f = h^{-1}gh$.*

2 Hyperbolic Sets and Anosov Diffeomorphisms

In this section, we denote M a C^1 manifold, U a non-empty open subset of M, $f : U \to f(U) \subset M$ a C^1 diffeomorphism, and $df : TM \to TM$ the differential of f. Several proofs require too much materials beyond the preliminaries we write before; therefore, for these proofs, we only give a sketch of proof or provide the references.

Definition 2.1 (Hyperbolic sets). *A compact, f-invariant subset $\Lambda \subset M$ is called hyperbolic if there are C^1 Riemannian metric, $\lambda \in (0, 1)$, $C > 0$ such that*

1. $T_\Lambda M = E^s \oplus E^u$
2. $df_x E^s(x) = E^s(f(x))$ *and* $df_x E^u(x) = E^u(f(x))$
3. $\|df_x^n v\| \leq C\lambda^n \|v\|$ *for all* $v \in E^s(x)$ *and* $n > 0$
4. $\|df_x^{-n} v\| \leq C\lambda^n \|v\|$ *for all* $v \in E^u(x)$ *and* $n > 0$

Proposition 2.2. *$E^s(x)$ and $E^u(x)$ continuously depend on x.*

Proof. We will prove for any converged sequence $\{x_n\}_{n\in\mathbb{N}}$ in Λ with the limit x we have $\lim_{n\to\infty} E^s(x_n) = E^s(x)$ and $\lim_{n\to\infty} E^u(x_n) = E^u(x)$.

Assuming that $\dim E^s(x_i) = k$, a constant, we let $w_{1,i}, w_{2,i}, ..., w_{k,i}$ be an orthonormal basis of $E^s(x_i)$. Let i tend to infinity, then we can get $w_1, ..., w_k$, an orthonormal subset with property $\|df_x^n w_j\| \leq C\lambda^n \|w_j\|$ for all $n \in \mathbb{N}$. Therefore, $\{w_1, ..., w_k\} \subset E^s(x)$, which implies $\lim_{n\to\infty} E^s(x_n) \subset E^s(x)$ and $\dim E^s(x) \geq k$.

Similarly, we can get $\dim E^u(x) \geq s - k$, where $\dim T_x M = s$.

Besides, by $s = \dim T_x M = \dim E^s(x) + \dim E^u(x) \geq s - k + \dim E^s(x)$, $k \geq \dim E^s(x)$. Therefore, $\dim E^s(x) = k$ and $w_1, ..., w_k$ form a basis for $E^s(x)$. It means $\lim_{n\to\infty} E^s(x_n) = E^s(x_0)$. Similarly, $\lim_{n\to\infty} E^u(x_n) = E^u(x)$. □

Definition 2.3 (Anosov diffeomorphisms). *A diffeomorphism $f : M \to M$ is called Anosov diffeomorphism if M is a hyperbolic set of f.*

We will use the following theorem to prove the structural stability of Ansov diffeomorphisms:

Theorem 2.4 (Shadowing Theorem)**.** *Let Λ be a hyperbolic set of $f : U \to f(U) \subset M$. Then there is an open set $O \subset U$ containing Λ and $\epsilon_0, \delta_0 > 0$, satisfying:*

$\forall \epsilon > 0, \exists \delta > 0$ *such that for any $g : O \to M$ with $\mathrm{dist}_1(g, f) < \epsilon_0$, any homeomorphism $h : X \to X$ of a topological space X, and any continuous map $\phi : X \to O$ with $\mathrm{dist}_0(\phi h, g\phi) < \delta$ there is a continuous map $\psi : X \to O$ with $\psi h = g\psi$ and $\mathrm{dist}_0(\phi, \psi) < \epsilon$.*

Moreover, if $\psi' h = g\psi'$ for some $\psi' : X \to O$ with $dist_0(\phi, \psi') < \delta_0$ then $\psi' = \psi$.

Sketch of proof. Refer to [1, p.566] for the details.

We will apply the contraction mapping principle in the proof. Notice that the desired map $\psi : X \to O$ is a fixed point of

$$F : C(X, O) \to C(X, M), \psi \to g\psi h^{-1}.$$

In order to keep $\mathrm{dist}_0(\phi, \psi) < \epsilon$, for sufficiently small $\theta > 0$, consider the map

$$\mathcal{A} : B_\theta(\phi) \to C_\phi(X, TM),$$

$$\text{given by } (\mathcal{A}\psi)(y) = \exp^{-1}_{\phi(y)} \psi(y),$$

where $\exp^{-1}_p$ is the inverse of exponential map of M at p, $B_\theta(\phi) = \{\psi \in C(X, O) : \mathrm{dist}_0(\phi, \psi) < \theta\}$, $C_\phi(X, TM) = \{v \in C(X, TM) : v(y) \in T_{\phi(y)}M, \forall y \in X\}$.

Then we define $F^\phi = \mathcal{A}F\mathcal{A}^{-1}$, which can be decomposed into linear and nonlinear parts, that is, $F^\phi(v) = dF^\phi_0 v + H(v)$. Now we define

$$T(v) = -(dF^\phi_0 - \mathrm{Id})^{-1}H(v).$$

T can be proved to be contracting on a sufficiently small neighborhood of ϕ. By contraction mapping principle, we obtain v, the fixed point of T. Now we find a fixed point of F, $\mathcal{A}^{-1}v$. □

Besides, we introduce a criterion for hyperbolicity.

Firstly, given a continuous direct sum decomposition $T_\Lambda M = E^s \oplus E^u$, that is for every $x \in \Lambda$ and $v \in T_xM$ we have $v = v^s + v^u$ where $v^s \in E^s(x)$ and $v^u \in E^u(x)$, and $E^s(x)$, $E^u(x)$ continuously depend on x.

For $\alpha > 0$, let us define

$$K^s_\alpha(x) = \{v \in T_xM : \|v^u\| \le \alpha\|v^s\|\},$$

$$K^u_\alpha(x) = \{v \in T_xM : \|v^s\| \le \alpha\|v^u\|\}.$$

They are called *stable and unstable cones of size* α, respectively.

Let Λ be a hyperbolic set of f.

Proposition 2.5. $\forall \alpha > 0$, $\exists \epsilon > 0$ *such that* $\forall x \in \Lambda$, *we get* $df_x(K^u_\alpha(x)) \subset \{0\} \cup \mathrm{int}(K^u_\alpha(f(x)))$ *and* $df^{-1}_{f(x)}(K^s_\alpha(f(x))) \subset \{0\} \cup \mathrm{int}(K^s_\alpha(x))$.

Proof. For $x \in \Lambda$, let $v = v^s + v^u \in K^u_\alpha(x)$. We will prove $\|df_x(v^s)\| < \alpha\|df_x(v^u)\|$.

Firstly, note that there exists a metric such that the hyperbolic set Λ is with the constant $C = 1$ and $\lambda \in (0, 1)$, which is called an *adapted metric*. So by the hyperbolicity $\|df_xv^s\| \le \lambda\|v^s\|$. Then by the definition of $K^u_\alpha(x)$ we have $\|df_xv^s\| \le \alpha\lambda\|v^u\|$. Applying the hyperbolicity again, $\|df_xv^s\| \le \alpha\lambda^2\|df_xv^u\| < \alpha\|df_xv^u\|$.

The second equation can be proved similarly. □

Proposition 2.6. $\forall \delta > 0$, $\exists \alpha > 0$ *such that* $\forall x \in \Lambda$, $\|df^{-1}_x v\| \le (\lambda + \delta)\|v\|$ *for* $v \in K^u_\alpha(x)$ *and* $\|df_xv\| \le (\lambda + \delta)\|v\|$ *for* $v \in K^s_\alpha(x)$.

Proof. For $x \in \Lambda$, let $v = v^s + v^u \in K^s_\alpha(x)$. Choose the adapted metric as the proposition above.

Notice that when $\|v\| = 1$, $\|df_xv^u\| \to 0$ as $\alpha \to 0$, by continuity. So for every $\delta > 0$, there exists an α such that $\|df_xv^u\| \le \delta$.

Then we have $\|df_xv\| \le \|df_xv^s\| + \|df_xv^u\| \le \lambda\|v\| + \delta\|v\| = (\lambda + \delta)\|v\|$.

The first inequality for $v \in K^u_\alpha(x)$ can be proved similarly. □

Proposition 2.7. *Let $f : U \to f(U) \subset M$ be a C^1 diffeomorphism and Λ be a compact invariant set for $f : U \to M$. If there are $\alpha > 0$, $\lambda \in (0, 1)$ and subspaces decomposition $T_\Lambda M = \widetilde{E}^s \oplus \widetilde{E}^u$ such that $\widetilde{E}^s(x)$ and $\widetilde{E}^u(x)$ continuously depend on x, and the stable cones K^s_α and unstable cones K^u_α of size α determined by the decomposition satisfy the following properties for any $x \in \Lambda$:*

1. *$df_x K^u_\alpha(x) \subset K^u_\alpha(f(x))$ and $df^{-1}_{f(x)} K^s_\alpha(f(x)) \subset K^s_\alpha(x)$,*
2. *$\|df_x v\| < \|v\|$ for $0 \neq v \in K^s_\alpha(x)$ and $\|df^{-1}_x v\| < \|v\|$ for $0 \neq v \in K^u_\alpha(x)$.*

Then Λ is a hyperbolic set for f.

Proof. Firstly by the compactness we have $\exists \lambda \in (0, 1)$ such that $\|df_x v\| \leq \lambda \|v\|$ for $v \in K^s_\alpha(x)$, and $\|df^{-1}_x v\| \leq \lambda \|v\|$ for $v \in K^u_\alpha(x)$. For $x \in \Lambda$, we define

$$E^s(x) = \bigcap_{n \geq 0} df^{-n}_{f^n(x)} K^s(f^n(x)),$$

$$E^u(x) = \bigcap_{n \geq 0} df^{n}_{f^{-n}(x)} K^u(f^{-n}(x)).$$

Then Λ is hyperbolic set of f with decomposition $T_\Lambda M = E^s \oplus E^u$, constant λ and $C = 1$. □

At the end of the section, we present an important concept, stable and unstable manifolds. Their denseness is highly relevant to the topological transitivity of Anosov diffeomorphisms, which will be shown in Section 4.

Definition 2.8 (Stable and unstable manifolds)**.** *For a hyperbolic set Λ of $f : U \to f(U) \subset M$ and every $x \in \Lambda$, we define stable manifolds of x by*

$$W^s(x) = \{y \in M : d(f^n(x), f^n(y)) \to 0 \text{ as } n \to \infty\}$$

and unstable manifolds of x by

$$W^u(x) = \{y \in M : d(f^{-n}(x), f^{-n}(y)) \to 0 \text{ as } n \to \infty\}.$$

Definition 2.9. *For an Anosov diffeomorphism $f : M \to M$ and any $\epsilon > 0$, we define*

$$W^s_\epsilon(x) = \{y \in M : d(f^n(x), f^n(y)) < \epsilon, \forall n \in \mathbb{N}_0\},$$

$$W^u_\epsilon(x) = \{y \in M : d(f^{-n}(x), f^{-n}(y)) < \epsilon, \forall n \in \mathbb{N}_0\}.$$

The stable and unstable manifolds theorem[2, p.121] have shown the existence of W^u and W^s. In this note, we do not need all of the results of the theorem. Therefore, a part of results are collected in the following proposition.

We denote the distances along the stable and the unstable manifolds by d^s and d^u respectively.

Proposition 2.10. *For an Anosov diffeomorphism $f : M \to M$, there are $\lambda \in (0, 1)$, $C_p > 0$, $\epsilon, \delta > 0$, and a splitting $T_x M = E^s(x) \oplus E^u(x)$ for all $x \in M$ such that:*

1. *$df_x(E^s(x)) = E^s(f(x))$, and $df_x(E^u(x)) = E^u(f(x))$;*
2. *$\forall v^s \in E^s(x)$, $\|df_x v^s\| \leq \lambda \|v^s\|$, and $\forall v^u \in E^u(x)$, $\|df^{-1}_x v^u\| \leq \lambda \|v^u\|$;*
3. *$\forall y \in W^s(x)$, $d^s(f(x), f(y)) \leq d^s(x, y)$, and $\forall y \in W^u(x)$, $d^s(f^{-1}(x), f^{-1}(y)) \leq d^u(x, y)$;*
4. *$f(W^s(x)) = W^s(f(x))$, and $f(W^u(x)) = W^u(f(x))$;*
5. *$T_x W^s(x) = E^s(x)$, and $T_x W^u(x) = E^u(x)$;*
6. *if $d(x, y) < \delta$, $W^s_\epsilon(x) \cap W^u_\epsilon(x) = \{p_{x,y}\}$; moreover, $p_{x,y}$ continuously depends on x,y, and $d^s(p_{x,y}, x) \leq C_p d(x, y)$, $d^u(p_{x,y}, y) \leq C_p d(x, y)$.*

Proof. It is a direct result from Hadamard-Perron theorem in [1, p.242] and Proposition 5.9.1 in [2, p.128]. □

3 Properties of Anosov Diffeomorphisms

3.1 Anosov diffeomorphisms are structurally stable

Definition 3.1 (Structurally stability). *A C^1 map $f : M \to M$ is called structurally stable if there exists a neighborhood U of f in the C^1 topology such that every $g \in U$ is topologically conjugate to f.*

Lemma 3.2. *Let Λ be a hyperbolic set of $f : U \to M$. There is an open set $U(\Lambda)$ containing Λ and $\epsilon_0 > 0$ such that if $K \subset U(\Lambda)$ is a compact invariant subset of a diffeomorphism $g : U \to M$ with $\mathrm{dist}_1(g, f) < \epsilon_0$, then K is a hyperbolic set of g.*

Proof. Notice that $E^s(x)$ and $E^u(x)$ continuously depends on x by Proposition 3.2; therefore, we can continuously extend the subspace decomposition to $T_{U(\Lambda)}M = E^s \oplus E^u$. By Proposition 3.5 and Proposition 3.6, the cones $K^s_\alpha(x)$ and $K^u_\alpha(x)$ with $x \in U(\Lambda)$ determined by the decomposition have the following properties:

1. $df_x K^u_\alpha(x) \subset K^u_\alpha(f(x))$ and $df^{-1}_{f(x)} K^s_\alpha(f(x)) \subset K^s_\alpha(x)$;
2. $\|df_x v\| \leq (\lambda + \delta)\|v\|$ for $0 \neq v \in K^s_\alpha(x)$ and $\|df^{-1}_x v\| \leq (\lambda + \delta)\|v\|$ for $0 \neq v \in K^u_\alpha(x)$.

And we have $\mathrm{dist}_1(f, g) = \mathrm{dist}_0(df, dg) < \epsilon_0$. For a sufficiently small ϵ_0 we can replace f by g.

That is, for $0 \neq v \in K^s_\alpha(x)$, $\|dg_x v\| \leq dist_0(df, dg)\|v\| + (\lambda+\delta)\|df_x v\| < \|v\|$. Similarly, $\|dg_x v\| < \|v\|$ for $0 \neq v \in K^u_\alpha(x)$, $dg_x K^u_\alpha(x) \subset K^u_\alpha(g(x))$ and $dg^{-1}_{g(x)} K^s_\alpha(g(x)) \subset K^s_\alpha(x)$.

Then by Proposition 3.7, the proof is finished if we let K be an compact invariant subset of $U(\Lambda)$. □

Corollary 3.3. *Anosov diffeomorphisms form an open set in* $\mathrm{Diff}^1(M)$.

Proof. Let $f : M \to M$ be an Anosov diffeomorphism. By Lemma 4.2, there is $\epsilon_f > 0$ such that for any diffeomorphism $g : M \to M$ with $\mathrm{dist}_1(g, f) < \epsilon_f$, g is an Anosov diffeomorphism. □

Theorem 3.4. *Let $\Lambda \subset M$ be a hyperbolic set of the diffeomorphism $f : U \to M$. Then for every open set $V \subset U$ containning Λ and every $\epsilon > 0$, there exists $\delta > 0$ such that if $g : V \to M$ with $\mathrm{dist}_1(f, g) < \delta$, there is a hyperbolic set K of g and a homeomorphism $\chi : K \to \Lambda$ such that $\chi g = f\chi$ and $\mathrm{dist}_0(\chi, \mathrm{Id}) < \epsilon$.*

Proof. By the Shadowing Theorem, for every $\epsilon > 0$ there exists $\delta > 0$ such that if we are given $X = \Lambda, h = f|_\Lambda, \phi = \mathrm{Id}_\Lambda$ and $g : V \to M$ is choosen to satisfy $\mathrm{dist}_1(f, g) < \delta$, there is a continuous map $\psi : \Lambda \to U$ with $\psi f|_\Lambda = g\psi$.

Set $K = \psi(\Lambda)$. By Lemma 4.2, there exist $\epsilon_0 > 0$ such that K is a hyperbolic set of g if $\mathrm{dist}_1(f, g) < \epsilon_0$. Therefore, we can let δ above be not greater than ϵ_0 to keep the hyperbolicity of K.

Now since K is hyperbolic set of g we can apply the Shadowing theorem again by taking $X = K$, $h = g|_K$, $\phi = \mathrm{Id}_K$. There is a continuous map $\psi' : K \to U$ with $\psi' g|_K = f|_\Lambda \psi'$.

By the equation above, we can get $\psi'\psi f|_\Lambda = f|_\Lambda \psi'\psi$ and $g|_K \psi\psi' = \psi\psi' g|_K$. By the uniqueness, we get $\psi\psi' = \mathrm{Id}_\Lambda$ and $\psi\psi' = \mathrm{Id}_K$. That is $\psi^{-1} = \psi'$.

Finally, let the homeomorphism $\chi : K \to \Lambda$ be ψ' to finish the proof. □

Corollary 3.5. *Anosov diffeomorphisms are structurally stable.*

Proof. Let $f : M \to M$ be an Anosov diffeomorphism. By Theorem 4.3, for every $\epsilon > 0$, there is $\delta > 0$ such that for any $g : M \to M$ with $\mathrm{dist}_1(f, g) < \delta$, we get a $h : M \to M$ with $\mathrm{dist}_0(h, \mathrm{Id}) < \epsilon$ and $hg = fh$. Therefore, f is structurally stable by Definition 4.1. □

3.2 Topological transitivity of Anosov diffeomorphisms

In this subsection, M is a connected compact Hausdorff smooth Riemannian manifold.

Lemma 3.6. *Let $f : M \to M$ be an Anosov diffeomorphism. Then* $\mathrm{Per}(f)$ *is dense in* $\mathrm{NW}(f)$.

Proof. It is sufficient to prove that for any $x \in \mathrm{NW}(f)$ and $\epsilon > 0$ there exists $p \in \mathrm{Per}(f)$ such that $d(x,p) < 2\epsilon$.

For every $\epsilon > 0$, choose $\delta \in (0,\epsilon)$ by the Shadowing theorem. Because $V = \{z \in M : d(z,x) < \delta/2\}$ is a neighborhood of $x \in \mathrm{NW}(f)$, there is a $n \in \mathbb{N}$ such that $f^n(V) \cap V$ is nonempty. Take a $z \in V \cap f^{-n}(V)$.

Let $h : \mathbb{Z}_n \to \mathbb{Z}_n$ by $h(m) = m+1$, $\phi : \mathbb{Z}_n \to M$ by $\phi(m) = z_m$ where $z_m = f^m(z)$. Since $d(z_{m+1}, f(z_m)) = 0$ for every $m < n-1$ in $\mathbb{Z}_n$ and $d(z_n, f(z_{n-1})) \le d(z,x) + d(x, f^n(z)) \le \delta$, by the Shadowing theorem, there exists $\{p_m\}_{m\in\mathbb{Z}_n}$ such that $p_{m+1} = f(p_m)$, $p_0 = f(p_{n-1})$ and $d(p_m, z_m) < \epsilon$.

We get $d(p_0,x) \le d(p_0,z) + d(z,x) < 2\epsilon$. ☐

Definition 3.7 (ϵ-dense)**.** *A subset $A \subset X$ is called to be ϵ-dense in a metric space (X,d) if $d(x,A) < \epsilon$ for every $x \in X$.*

Theorem 3.8. *Let $f : M \to M$ be an Anosov diffeomorphism. Then the following are equivalent:*

1. $\mathrm{NW}(f) = M$,
2. *Every unstable manifold is dense in M,*
3. *Every stable manifold is dense in M,*
4. *f is topologically transitive,*
5. *f is topologically mixing.*

Proof. $1 \Rightarrow 2$: We need to prove that $\forall\epsilon > 0$, $\forall x \in M$, the unstable manifold $W^u(x)$ is ϵ-dense in M.

Firstly, for any $\epsilon > 0$, we construct a $\epsilon/4$-dense set with finite elements in $\mathrm{Per}(f)$. By Lemma 4.6, $\mathrm{Per}(f)$ is dense in M, which implies $M = \bigcup_{x\in\mathrm{Per}(f)} B(x,\epsilon/4)$. Therefore, by compactness of M, there are N elements in $\mathrm{Per}(f)$, $\{x_i\}_{i=1,2,\ldots,N}$, such that $M = \bigcup_{i=1}^{N} B(x_i,\epsilon/4)$. That is, $\forall x \in M$, $\exists i$ such that $\mathrm{dist}(x,x_i) < \epsilon/2$ and $x_i \ne x$.

Let the product of their periods be P and define $g = f^P$. We prove that g and f have the same unstable manifolds, so we can use the unstable manifolds of g to instead of those of f. Let $\widetilde{W}^u(x) = \{y \in M : d(g^n(x), g^n(y) \to 0 \text{ as } n \to \infty\}$ be the unstable manifolds of g at x. Obviously, $W^u(x) \subset \widetilde{W}^u(x)$. Because f is C^1, for every $n \in \mathbb{N}$, $d(f^{nP+1}(x), f^{nP+1}(y)) < d(f^{nP}(x), f^{nP}(y))$. Then $\widetilde{W}^u(x) \subset W^u(x)$.

And we need to notice the following lemma:

Lemma 3.9. *$\exists q \in \mathbb{N}$ such that if for some y,x_i,x_j such that $\mathrm{dist}(W^u(y), x_i) < \epsilon/2$ and $\mathrm{dist}(x_i,x_j) < \epsilon/2$, then we have $\mathrm{dist}(g^{nq}(W^u(y)), x_i) < \epsilon/2$ and $\mathrm{dist}(g^{nq}(W^u(y)), x_j) < \epsilon/2$.*

Proof. We choose appropriate ϵ to make sure that $W^u(y) \cap W^s_e(x_i) \ne \emptyset$ for a sufficiently small e by Proposition 3.10.

Then let $z \in W^u(y) \cap W^s_e(x_i)$. Because $z \in W^s(x_i) = \bigcup_{n=0}^{\infty} f^{-n} W^s_e(f^n(x_i))$, there is a t_0 such that $\mathrm{dist}(g^t(z), x_i) < \epsilon/2$ for all $t \ge t_0$ (We need note that $g^t(x_i) = x_i$). Then we have $\mathrm{dist}(g^t(W^u(z)), x_j) < \epsilon$ by the triangle inequality.

We can gain a $w \in W^u(g^t(z)) \cap W^s_{e'}(x_j)$ for a sufficiently small e' by Proposition 3.10, for the distance between $g^t(W^u(z))$ and x_j is small enough. Because $w \in W^s(x_j)$, there is a s_0 such that $\mathrm{dist}(g^\tau(w), x_j) < \epsilon/2$ for all $\tau \ge s_0$.

Finally, we make $q = s_0 + t_0$ to finish the lemma. ☐

Since the set $\{x_i\}_{i=1,2,\ldots,N}$ is $\epsilon/4$-dense, $\forall y \in M$, $\exists x_s$ such that $\mathrm{dist}(g^{nq}(W^u(y)), x_s) < \epsilon/2$. And for any x_t there is a chain in $\{x_i\}_{i=1,2,\ldots,N}$ which connects x_t and x_s and the distance between two consecutive points less than $\epsilon/2$, because M is compact and connected. Note that the length of the chain will not be larger than N.

Therefore, for every $z \in M$, if we choose i such that $\mathrm{dist}(z,x_i) < \epsilon/2$ and let $y = g^{-Nq}(x)$ for any x, we can obtain $\mathrm{dist}((W^u(x)), z) \le \mathrm{dist}((W^u(x)), x_i) + \mathrm{dist}(x_i, z) < \epsilon$. It means $W^u(x)$ is ϵ-dense in M for any $x \in M$ and $\epsilon > 0$.

$1 \Rightarrow 3$: Similarly.

$2 \Rightarrow 5$: We need to prove that $\exists N$, $\forall n > N$, $f^n(U) \cap V \ne \emptyset$ for any non-empty sets U and V, by the definition of topoligically mixing.

Let us choose $x, y \in M$ and $\delta > 0$ such that $W^u_\delta(x) \subset U$ and $B(y,\delta) \subset V$. Notice that $f^n(W^u_\delta(x)) \subset f^n(U)$, for any $n \in \mathbb{N}$.

Lemma 3.10. *If for every $x \in M$, $W^u(x)$ is dense in M, then $\forall \delta > 0$, $\exists R = R(\delta) > 0$ such that every ball of radius R in every unstable manifold is ϵ-dense in M.*

Proof. Because $W^u(x) = \bigcup_R W^u_R(x)$ is dense in M, there is R which depends on x and $W^u_R(x)$ is $\epsilon/2$-dense in M. Since W^u is a continuous foliation, we get $\delta(x)$ such that $W^u_R(y)$ is ϵ-dense, $\forall y \in B(x, \delta)$.

Now we have $M = \bigcup_{x \in M} B(x, \delta(x))$. By compactness we find a finite collection for those balls $B(x, \delta(x))$. Then choose the maximal $R(x)$ for those balls to finish the lemma. □

Notice that $\exists N$, $\forall n > N$, $W^u_R(f^n(x)) \subset f^n(W^u_\delta(x))$.

By the lemma above, for $n > N$, $B(y, \delta) \cap W^u_R(f^n(x)) \neq \emptyset$, which means $V \cap f^n(U) \neq \emptyset$. Hence f is topologically mixing.

$3 \Rightarrow 5$: Similarly.

$5 \Rightarrow 4 \Rightarrow 1$: It is obvious by their definitions. □

4 Examples

In this section, we firstly introduce a procedure to construct an Anosov diffeomorphism on nilmanifolds. Then we will follow the procedure to give several examples of Anosov diffeomorphisms.

4.1 A brief introduction to nilmanifolds

Before giving the detail of the construction, we need to give a brief introduction to nilmanifolds.

Definition 4.1 (Lie groups)**.** *A group G with a smooth manifold structure is called Lie group, if the following group operations are smooth:*

$$(a, b) \to ab,$$

$$a \to a^{-1},$$

where $a, b \in G$.

Definition 4.2 (Lie group homomorphisms)**.** *Given Lie group G and H, A map $g : G \to H$ is called a Lie group homomorphism if it is a smooth map and also a group homomorphism from G to H.*

It is called a Lie group isomorphism if it is also a diffeomorphism.

A Lie group isomorphism $g : G \to G$ is called a Lie group automorphism.

And in the rest of the subsection, we use G to represent a Lie group.

By the definition above, we can define the following smooth map, left transition for every $g \in G$:

$$L_g(h) = gh.$$

In fact, $L_g : G \to G$ is a diffeomorphism with smooth inverse $L_{g^{-1}}$. Therefore, for an arbitrary smooth vector field X, we can obtain $(L_g)_*X$, a smooth vector field, which is defined as following:

$$((L_g)_*X)_h = (dL_g)_{g^{-1}h}(X_{g^{-1}h}),$$

where $h \in G$.

Now we give a brief introduction to Lie algebras.

Definition 4.3 (Lie algebra)**.** *Let L be a vector space. A map from $L \times L$ to L denoted $(X, Y) \to [X, Y]$ is called a Lie bracket on L if*

1. *the map is bilinear;*
2. *$\forall X \in L, [X, X] = 0$;*
3. *$\forall X, Y, Z \in L, [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.*

The last condition is called Jacobi identity.

The vector space L with a Lie bracket is called a Lie algebra.

Definition 4.4 (Lie algebra homomorphisms)**.** *A vector space homomorphism* $\phi : L \to L'$ *is called a Lie algebra homomorphism, if* $\phi([X,Y]) = [\phi(X), \phi(Y)]$.

A vector space isomorphism $\phi : L \to L'$ *is called a Lie algebra isomorphism if* $\phi([X,Y]) = [\phi(X), \phi(Y)]$.

A Lie algebra isomorphism $\phi : L \to L$ *is called a Lie algebra automorphism.*

We call a Lie algebra automorphism is hyperbolic, if its eigenvalues are away from the unit circle.

Definition 4.5 (Structure constants of Lie algebras)**.** *Let* L *be a Lie algebra with a basis* $X_1, X_2, ...X_n$. *We have*

$$[X_i, X_j] = \sum_{k=1}^{n} a_{ij}^k X_k$$

for any $i, j \leq n$. *We call those* a_{ij}^k *are structure constants of* L.

Example 4.6 (Lie algebra of G)**.** *Now we define a Lie algebra associated with Lie group* G. *Firstly we let*

$$\mathrm{Lie}(G) = \{X \text{ is a smooth vector field on } G : (L_g)_* X = X, \forall g \in G\}.$$

And for any smooth vector fields X, Y *on* G, *define a smooth vector field* $[X,Y]$ *by*

$$[X,Y]f = XYf - YXf,$$

for every smooth map $f : G \to G$. *(Note that* $[X,Y]$ *is a smooth vector field although we have not shown that.)*

$\mathrm{Lie}(G)$ *is called Lie algebra of* G, *and the map* $(X,Y) \to [X,Y]$ *is a Lie bracket on* $\mathrm{Lie}(G)$.

We define $[x,y] = x^{-1}y^{-1}xy$. Notice the lower central series of G:

$$\gamma_1(G) = G,$$

$$\gamma_2(G) = [\gamma_1(G), G],$$

$$...$$

$$\gamma_{k+1}(G) = [\gamma_k(G), G].$$

Definition 4.7 (Nilpotent Lie group)**.** G *is called a nilpotent Lie group, if* $\exists n \in \mathbb{N}$ *such that* $\gamma_n(G) = 1$.

Definition 4.8 (c-step nilpotent Lie group)**.** G *is called a c-step nilpotent Lie group, if* $\gamma_c(G) \neq 1$ *and* $\gamma_{c+1}(G) = 1$.

Definition 4.9 (Lattice)**.** N *is a simply-connected and nilpotent Lie group. Let* Γ *be a discrete subgroup of* G. *It is called a lattice in* G *if* G/Γ *is a compact quotient space.*

Note that the quotient space G/Γ must be a smooth manifold by Theorem 21.29 in Lee's book. Some nilpotent Lie groups do not admit any lattices. Therefore, we need the following criterion:

Proposition 4.10 (Maltsev's criterion)**.** *A nilpotent Lie group* N *admits some lattices if and only if all of the structure constants of* $\mathrm{Lie}(N)$ *are in* $\mathbb{Q}$.

Proof. Refer to Theorem 2.12 in Raghunathan's book. □

Definition 4.11 (Nilmanifolds)**.** *A differential manifold* N/Γ *is called a nilmanifold, if* N *is a simply-connected nilpotent Lie group and* Γ *is a lattice in* N.

Example 4.12 ($\mathbb{R}^n$ as a Lie group)**.** $\mathbb{R}^n$ *is a Lie group with the group operation* $+$. *Obviously,* $\mathbb{R}^n$ *is an abelian group, so a nilpotent Lie group.*

Moreover, it is well-known that $\mathbb{R}^n$ *is simply-connected.*

Example 4.13 ($\mathbb{T}^n$ as a Lie group)**.** *It is easy to verify that* $\mathbb{Z}^n$ *is a normal subgroup of* $\mathbb{R}^n$. *And it is also a closed Lie subgroup of* $\mathbb{R}^n$, *for it is discrete subgroup. Therefore,* $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ *is a Lie group.*

And $\mathbb{T}^n \cong \mathbb{S}^1 \times \mathbb{S}^1 \times ... \times \mathbb{S}^1$, *which implies compactness. So* $\mathbb{Z}^n$ *is a lattice of* $\mathbb{R}^n$.

4.2 Example: An Anosov diffeomorphism on tori

Now we can introduce the procedure to construct an Anosov diffeomorphism on a nilmanifold. Let N be a simply-connected nilpotent Lie group, Γ be a lattice in N, $f : N \to N$ be a Lie group automorphism of N such that $f(\Gamma) = \Gamma$, and $df_{\mathrm{Id}} : T_{\mathrm{Id}}N \to T_{\mathrm{Id}}N$, the Lie algebra automorphism induced by f, is hyperbolic. Obviously, the procedure induces a Anosov diffeomorphism $\widetilde{f} : N/\Gamma \to N/\Gamma$ on a nilmanifold.

The procedure is reasonable because Smale pointed out that if a Lie algebra admits a hyperbolic Lie algebra automorphism then it must be nilpotent, which forces N to be nilpotent and Γ to be a uniform discrete subgroup.

It is necessary to point out that there are other methods to construct Anosov diffeomorphisms. Readers can refer to F. T. Farrell and L. E. Jones's paper *Anosov diffeomorphisms constructed from* $\pi_1 \mathrm{Diff} S^n$. Their paper also show that not every smooth manifold which admits an Anosov diffeomorphism is diffeomorphism to an infranilmanifold. However, we still do not know whether every smooth manifold which admits an Anosov diffeomorphism is homeomorphism to an infranilmanifold.

Then we will construct an Anosov diffeomorphisms on $\mathbb{T}^2$ as an example.

Example 4.14. *As we show in Example 5.12 and Example 5.13,* $\mathbb{R}^2$ *is a simply-connected nilpotent Lie group and* $\mathbb{Z}^2$ *is a lattice in* $\mathbb{R}^2$*.*

Firstly, we define a linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$ *given by the matrix* $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$*. Because* $\det L = 1$*, we have* $L(\mathbb{Z}^2) = \mathbb{Z}^2$*. And we denote the eigenvalues of* L *by* λ *and* $1/\lambda$*, where the value of* λ *is* $\frac{3-\sqrt{5}}{2}$*.*

The map L *induces a diffeomorphism* $F_L : \mathbb{T}^2 \to \mathbb{T}^2$ *by*

$$F_L(x, y) = (2x + y, x + y) (\mathrm{mod}\ 1),$$

where $x, y \in \mathbb{R}/\mathbb{Z}$*.*

Now we prove that F_L is an Anosov diffeomorphism.

Proposition 4.15. F_L *is an Anosov diffeomorphism.*

Proof. For any $p \in \mathbb{T}^2$, the differential of F_L at p is a vector space isomorphism $(dF_L)_p : T_p\mathbb{T}^2 \to T_{F_L(p)}\mathbb{T}^2$. The matrix of $(dF_L)_p$ in term of the coordinate basis is $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

Then we have a dirct sum decomposition, $T_p\mathbb{T}^2 = E^s_p \oplus E^u_p$, where E^s_p, E^u_p are eigenspaces associated with $\lambda, 1/\lambda$, respectively. It is easy to verify that $\{E^s_p\}_{p\in\mathbb{T}^2}$ $(\{E^u_p\}_{p\in\mathbb{T}^2})$ form the stable (unstable) distribution of F_L. Therefore, F_L is an Anosov diffeomorphism. □

Finally, we prove that F_L is topologically mixing. It is sufficient to prove $\mathrm{Per}(F_L)$ is dense in $\mathbb{T}^2$. By Lemma 4.6 and Theorem 4.8, F_L is topologically mixing.

Proposition 4.16. *The periodic points of* F_L *are dense in* $\mathbb{T}^2$*.*

Proof. We claim that all points with a rational coordinate are periodic points. The claim implies the proposition is true. Now we prove the claim. Firstly, we let $p = (s/q, t/q) \in \mathbb{T}^2$, an arbitrary point with a rational coordinate.

Consider the set of rational points on $\mathbb{T}^2$ with denominator q. It is a finite set with q^2 element and contains $\{F^n_L(s/q, t/q)\}_{n\geq 0}$, which means that $\exists M, N \in \mathbb{N}$ such that $F^M_L(s/q, t/q) = F^N_L(s/q, t/q)$.

Now recall that F_L is a diffeomorphism. The proof is finished. □

4.3 Example: An Anosov diffeomorphism on a nontoral manifold

In this subsection, we firstly introduce the Heisenberg group and use it to construct another example of Anosov diffeomorphism.

We define the Heisenberg group

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}$$

with the matrix multiplication.

The Lie algebra of H is given by

$$\text{Lie}(H) = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}$$

with generators

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now we can begin our construction. We let $G = H \times H = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \middle| A, B \in H \right\}$, which is a simply-connected nilpotent Lie group. Moreover, the basis of Lie(G) contains

$$X_1 = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}, Y_1 = \begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix}, Z_1 = \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix},$$

$$X_2 = \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix}, Y_2 = \begin{pmatrix} 0 & 0 \\ 0 & Y \end{pmatrix}, Z_2 = \begin{pmatrix} 0 & 0 \\ 0 & Z \end{pmatrix}.$$

For any $\lambda > 1$, we define $F : \text{Lie}(G) \to \text{Lie}(G)$ given by

$$\begin{aligned} F(X_1) &= \lambda X_1, \quad F(X_2) = \lambda^{-1} X_2, \\ F(Y_1) &= \lambda^2 Y_1, \quad F(Y_2) = \lambda^{-2} Y_2, \\ F(Z_1) &= \lambda^3 Z_1, \quad F(Z_2) = \lambda^{-3} Z_2, \end{aligned}$$

which is a hyperbolic Lie algebra automorphism on Lie(G).

By Proposition 5.10, G admits some lattices because $[X, Y] = Z$ and other Lie brackets of generators are zero. Actually we can give a lattice Γ defined by $\exp_{\text{Id}}(\gamma)$ with

$$\gamma = \left\{ \begin{pmatrix} A & 0 \\ 0 & \sigma(A) \end{pmatrix} \in \text{Lie}(G) \middle| A = \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \text{ for } x, y, z \text{ are algebraic integer in } \mathbb{K} \right\},$$

where $\mathbb{K} = \mathbb{Q}(\sqrt{3}) = \{a + b\sqrt{3} : a, b \in \mathbb{Q}\}$, $\sigma : a + b\sqrt{3} \to a - b\sqrt{3}$.

Notice we have the following decomposition:

$$\text{Lie}(G) = \text{Lie}(G^u) \oplus \text{Lie}(G^s),$$

where $G^u = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \middle| A \in \text{Lie}(H) \right\}$, and $G^s = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \middle| B \in \text{Lie}(H) \right\}$.

By Lie group and Lie algebra theory, there is a unique automorphism $f : G \to G$ with $df|_{\text{Id}} = F$ and $f(\Gamma) = \Gamma$. It induces an Anosov diffeomorphism of G/Γ.

Proposition 4.17. *Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear Lie group automorphism on $\mathbb{R}^2$ with integer elements. If* $\det L = 1$ *and the absolute values of eigenvalues of L are not* 1*, then L induced an Anosov diffeomorphism F_L on $\mathbb{T}^2$.*

Proof. Let $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the matrix of the linear Lie group automorphism on $\mathbb{R}^2$, where a, b, c, d are integers. We can gain the result that $L(\mathbb{Z}^2) = \mathbb{Z}^2$ and assume that the eigenvalues of L are λ and $1/\lambda$ with $|\lambda| < 1$, because $\det L = 1$.

Note that $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Then the automorphism on $\mathbb{T}^2$ induced by L can be writen as:

$$F_L(x, y) = (ax + by, cx + dy)(\text{mod } 1).$$

We need to prove F_L is an Anosov diffeomorphism on $\mathbb{T}^2$.

Define $E_\lambda = \{v \in \mathbb{R}^2 : Lv = \lambda v\}$ and $E_{1/\lambda} = \{v \in \mathbb{R}^2 : \lambda Lv = v\}$. Then on E_λ we have $\|L^n v\| = \lambda^n \|v\|$ and on E_λ we have $\|L^{-n} v\| = \lambda^n \|v\|$, where the norm is induced by Euclidean metric.

Because the tangent bundle of the tori $T\mathbb{T}^2 \cong \mathbb{T}^2 \times \mathbb{R}^2$ and $\mathbb{R}^2 = E_\lambda \oplus E_{1/\lambda}$, the tangent space of $\mathbb{T}^2$ at any point can be splitted into stable subspace E_λ and the unstable subspace $E_{1/\lambda}$.

Then by $L(E_\lambda) = E_\lambda$ and $L(E_{1/\lambda}) = E_{1/\lambda}$, we finish the proof. □

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