The Solution to a Special Upper-Triangular Toeplitz System

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1 Introduction

A Toeplitz matrix is a square matrix with constant diagonals, which means that each element along a diagonal has the same value. Formally, an $n \times n$ Toeplitz matrix T can be defined as follows:

$$T = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & \dots & t_{-(n-1)} \\ t_1 & t_0 & t_{-1} & \dots & t_{-(n-2)} \\ t_2 & t_1 & t_0 & \dots & t_{-(n-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} & \dots & t_0 \end{bmatrix}$$

where t_k denotes the constant value on the k-th diagonal. In this note, we consider a special form of Toeplitz matrix and present the solution to the linear system constructed by this matrix.

2 The Main Result

Theorem 2.1. *Suppose the sequence* $\left\{(\mathsf{a}_n, \mathsf{b}_n)\right\}_{n=1}^{N}$ *satisfies the following recursion:*

$$\begin{aligned} \mathsf{a}_N &= \mathsf{C}_1; \\ \mathsf{a}_n &= \mathsf{C}_1 + \mathsf{C}_2 \sum_{i=0}^{N-n} \mathsf{C}_3^{i+1} \mathsf{a}_{n+1+i}. \end{aligned}$$

Then for all n,

$$\begin{aligned} \mathsf{a}_n &= \alpha \frac{1-\beta^{N-n+1}}{1-\beta} + \beta^{N-n+1} \mathsf{a}_N \\ &= \frac{\alpha}{1-\beta} + (\mathsf{C}_1 - \frac{\alpha}{1-\beta})\beta^{N-n+1} \\ &= \frac{\mathsf{C}_1(1-\mathsf{C}_3)}{1-(1+\mathsf{C}_2)\mathsf{C}_3} + \frac{\mathsf{C}_1\mathsf{C}_2\mathsf{C}_3}{(1+\mathsf{C}_2)\mathsf{C}_3 - 1}(1+\mathsf{C}_2)^{N-n+1}\mathsf{C}_3^{N-n+1}. \end{aligned}$$

Proof. We re-write the recursion in matrix form.

$$\begin{bmatrix} \mathsf{a}_1 \\ \mathsf{a}_2 \\ \vdots \\ \mathsf{a}_N \end{bmatrix} = \mathsf{C}_1 \mathbf{1}_H + \mathsf{C}_2 \begin{bmatrix} 0 & \mathsf{C}_3 & \mathsf{C}_3^2 & \dots & \mathsf{C}_3^{N-1} \\ 0 & 0 & \mathsf{C}_3 & \dots & \mathsf{C}_3^{N-2} \\ 0 & 0 & 0 & \dots & \mathsf{C}_3^{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \mathsf{a}_1 \\ \mathsf{a}_2 \\ \vdots \\ \mathsf{a}_N \end{bmatrix}.$$

We start from investigating two consecutive rows of this equality;

$$\begin{aligned} \mathsf{a}_n &= \mathsf{C}_1 + \mathsf{C}_2 \sum_{i=0}^{N-n-1} \mathsf{C}_3^{i+1} \mathsf{a}_{n+1+i}; \\ \mathsf{a}_{n+1} &= \mathsf{C}_1 + \mathsf{C}_2 \sum_{i=1}^{N-n-1} \mathsf{C}_3^{i} \mathsf{a}_{n+1+i}. \end{aligned}$$

It solves the relation between a_n and a_{n+1}:

$$\mathsf{a}_n - \mathsf{C}_3\mathsf{a}_{n+1} = \mathsf{C}_1 - \mathsf{C}_3\mathsf{C}_1 + \mathsf{C}_2\mathsf{C}_3\mathsf{a}_{n+1};$$

or it can be re-written as

$$\mathsf{a}_n = \mathsf{C}_1(1 - \mathsf{C}_3) + (1 + \mathsf{C}_2)\mathsf{C}_3\mathsf{a}_{n+1}.$$

Let $\alpha = \mathsf{C}_1(1 - \mathsf{C}_3)$ and $\beta = (1 + \mathsf{C}_2)\mathsf{C}_3$. We can resolve this recursion to a_N:

$$\begin{aligned}
\mathsf{a}_n &= \alpha\frac{1 - \beta^{N-n+1}}{1 - \beta} + \beta^{N-n+1}\mathsf{a}_N \\
&= \frac{\alpha}{1 - \beta} + (\mathsf{C}_1 - \frac{\alpha}{1 - \beta})\beta^{N-n+1} \\
&= \frac{\mathsf{C}_1(1 - \mathsf{C}_3)}{1 - (1 + \mathsf{C}_2)\mathsf{C}_3} + \frac{\mathsf{C}_1\mathsf{C}_2\mathsf{C}_3}{(1 + \mathsf{C}_2)\mathsf{C}_3 - 1}(1 + \mathsf{C}_2)^{N-n+1}\mathsf{C}_3^{N-n+1}
\end{aligned}$$

□

Moreover, if $n = 1$ and $\mathsf{C}_3 = 1 + \frac{1}{N}$, we have the following special form:

$$\mathsf{a}_1 = \frac{\mathsf{C}_1}{\mathsf{C}_2 + 1 + \mathsf{C}_2 N}\left[(1 + \frac{1}{N})^N(1 + \mathsf{C}_2)^N - 1\right].$$

This result indicates that unless C_2 is very small, the value a_1 can grow exponentially.