

Anosov diffeomorphisms and their properties

Student: Ma Shaocong; Supervisor: Gao Rui

Abstract

In this thesis, we mainly focus on the properties of Anosov diffeomorphisms: structural stability and topological transitivity of Anosov diffeomorphisms. Besides, we also pay attention on the classification of nilmanifolds which admit Anosov diffeomorphisms. At the beginning of the thesis, we introduce some basic concepts necessary to the topic such as metric spaces, Riemannian manifolds and topological dynamics. Then we mention some basic results in hyperbolic dynamics including the shadowing theorem and the stable and unstable manifolds theorem. Using these tools, we prove that every Anosov diffeomorphism is structurally stable, and build some equivalent condition of topological transitivity of Anosov diffeomorphisms. Finally, we present several examples of Anosov diffeomorphisms.

Keywords: Anosov diffeomorphisms; structural stability; topological transitivity; nilmanifolds

1 Introduction

A diffeomorphism from a compact manifold to itself is called Anosov if the manifold is a hyperbolic set of the diffeomorphism. The concept was introduced by a Soviet and Russian mathematician, Dmitri Victorovich Anosov, who proved that Anosov diffeomorphisms are structurally stable and form an open subset of the space of C^1 diffeomorphisms with the C^1 topology [4].

Anosov diffeomorphisms play an important role in the theory of hyperbolic dynamics and have been studied for more than a half century, but there are still several open problems left to be solved. Firstly, whether every Anosov diffeomorphism is transitive is not known, although it is considered as highly probable [2].

Besides, classifying manifolds that admit Anosov diffeomorphisms is a main topic in this field, which is raised in Smale's paper [3]. Many results about the problems were proven. For example, the flat case was solved, due to H. Poteaux [22]. And for the case of nilmanifolds, in 2000 Wim Malfait gave a complete answer for the nilmanifolds of dimension at most 6 [19]; in 2008 Jorge Lauret and Cynthia E. Will extended the result to dimension 8 [18]. And for nilmanifolds modeled on a free c -step nilpotent Lie group with n generators, they admit an Anosov diffeomorphism if and only if $n > c$ [21]. In 2015, the result of [21] is generalized to the case of infra-nilmanifolds [23]. The problem can be also considered equivalent to the classification of Anosov Lie algebra [20].

Moreover, it is also conjectured that every Anosov diffeomorphism is equivalent to a hyperbolic automorphism on an infranilmanifold in topological sense [7]. For the simplest case, A. Avez proved that every Anosov diffeomorphism on \mathbb{T}^2 is topologically conjugate to an algebraic one [8]. And in 1969, under a conjecture that the non-wondering points are equal to the whole tori, J. Franks gave a more general result: Anosov diffeomorphisms on \mathbb{T}^n is topologically conjugate to hyperbolic toral automorphisms [6]. Then in 1974, A. Manning proved that on infranilmanifolds all Anosov diffeomorphisms are topologically equivalent to hyperbolic automorphisms [5]. In addition, S. E. Newhouse pointed out that Anosov diffeomorphisms with codimension 1 is topologically conjugate to a toral automorphism [9]. We also notice that some partial results have been studied [10–13].

Finally, we collect a lot of elementary results in the field of hyperbolic dynamics and list several basic properties of Anosov diffeomorphisms in this thesis, which are the foundations of the results we list above. In the future, we will pay more attention on hyperbolic dynamics, nilpotent Lie groups and Lie algebras, infra-nilmanifolds, and other fields related to Anosov diffeomorphisms.

Structure of the thesis. In this thesis, we introduce some basic properties of Anosov diffeomorphisms and give several classical examples of Anosov diffeomorphisms. Section 2 contains several necessary definitions and propositions which would be used in the rest of the thesis. In Section 3 we state some basic results in the hyperbolic dynamics. In Section 4 we list and prove the basic properties of Anosov diffeomorphisms, and in Section 5 we firstly provide some necessary mathematical background, then construct several examples of Anosov diffeomorphisms.

2 Preliminaries and notations

Although we have attempted to limit the amount of prerequisites as minimal as possible, there are still a large number of definitions necessary to our statement. Therefore, in this section, we introduce some definitions which would be used. Any missing definitions can be found at the reference books, *Introduction to the Modern Theory of Dynamical Systems* [1] and *Introduction to Dynamical Systems* [2].

We denote all of positive integers as N and all of non-negative integers as N_0 in this thesis.

2.1 Metric Spaces

Definition 2.1 (Metrics). *Let X be a set. A map $d : X \times X \rightarrow [0, \infty)$ is called a metric on X if*

1. $d(x, y) = d(y, x)$,
2. $d(x, y) = 0$ if and only if $x = y$,
3. $d(x, y) + d(y, z) \geq d(x, z)$.

(X, d) is called a metric space, where d is a metric and X is a topological space with topology induced by d . We also denote it by X when the metric is indicated.

Definition 2.2 (Complete metric space). *Let (X, d) be a metric space. A sequence $\{x_k\}_{k=1,2,\dots}$ in X is called a Cauchy sequence if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n, m > N$ we have $d(x_n, x_m) < \epsilon$.*

We call a metric space X is complete, if every Cauchy sequence in X converges in M .

Example 2.3 (The space of continuous maps). *Let X be a compact topological space, Y be a metric space with a metric d , and $C(X, Y)$ be the set of all continuous maps from X to Y .*

Define a metric dist_0 on $C(X, Y)$ by

$$\text{dist}_0(f, g) = \min\{1, \sup_{x \in X} \max\{d(f(x), g(x))\}\}$$

for $f, g \in C(X, Y)$.

Note that if Y is a complete metric space, then $C(X, Y)$ is complete as well, because its topology is as same as that induced by the uniform metric when X is compact. Refer to [2, p.140].

2.2 Riemannian manifolds

In this subsection, we give a brief introduction to Riemannian manifolds.

Definition 2.4 (C^k Riemannian manifolds). *A C^k Riemannian metric is a family of positive definite symmetric bilinear form $\{\langle \cdot, \cdot \rangle_p\}_{p \in M}$ defined on the tangent space $T_p M$ with the following property: For any C^k vector fields X and Y , the map $p \rightarrow \langle X_p, Y_p \rangle_p$ is C^k .*

A C^k Riemannian manifold is a C^k manifold with a C^k Riemannian metric.

When $k = \infty$, we call it a smooth Riemannian manifold.

In this thesis, we mainly focus on the smooth case.

Let $x = (x^1, \dots, x^d)$ be local coordinates. The metric can be represented by a positive definite, symmetric matrix

$$(g_{ij}(x))_{i,j=1,\dots,d},$$

where $g_{ij}(x)$ is smooth. And we denote the inverse of the matrix by $(g^{ij}(x))_{i,j=1,\dots,d}$.

For every $v \in T_p M$, we define $\|v\| = (\langle v, v \rangle_p)^{1/2}$. Let $[a, b]$ be a closed interval in \mathbb{R} , and $\gamma : [a, b] \rightarrow M$ be a smooth curve. We define the energy of γ by

$$E(\gamma) = \frac{1}{2} \int_a^b \left\| \frac{d\gamma}{dt} \right\|^2 dt.$$

We now rewrite it in local coordinates:

$$E(\gamma) = \frac{1}{2} \int_a^b g_{ij}(x(\gamma(t))) \dot{x}^i(t) \dot{x}^j(t) dt,$$

where the local coordinates of $\gamma(t)$ is $x(\gamma(t)) = (x^1(\gamma(t)), \dots, x^d(\gamma(t)))$, and $\dot{x}^i(t) = \frac{d}{dt}(x^i(\gamma(t)))$.

Then the Euler-Lagrange equations for the energy E are

$$\ddot{x}^i(t) + \Gamma_{jk}^i(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0, \quad i = 1, \dots, d$$

with the Christoffel symbols Γ_{jk}^i defined by

$$\Gamma_{jk}^i = \frac{1}{2}g^{il}(g_{jl,k} + g_{kl,j} - g_{jk,l}),$$

where $g_{jl,k} = \frac{\partial}{\partial x^k}g_{jl}$.

Definition 2.5 (Geodesic). *For a Riemannian manifold M , a differentiable curve $\gamma : [a, b] \rightarrow M$ is called a geodesic if it satisfies the Euler-Lagrange equations for the energy $E(\gamma)$.*

It is well-known that the geodesics are the shortest curves between two points which are sufficiently close and there always exist geodesics on compact manifolds. Moreover, for a compact Riemannian manifold M , any $p \in M$, $v \in T_pM$, there is a unique geodesic

$$c_v : (-\infty, +\infty) \rightarrow M$$

with $c(0) = p$, $\dot{c}(0) = v$; and c_v continuously depends on p and v . Refer to [24] for more details.

Definition 2.6 (Exponential map). *Let M be a compact Riemannian manifold, $p \in M$. We define the exponential map*

$$\exp_p : T_pM \rightarrow M, \quad \text{by } v \rightarrow c_v(1).$$

It is necessary to point out when M is not compact, the exponential map \exp_p may not be defined on the whole of T_pM . Fortunately, in this thesis, we only focus on the compact case, so \exp_p is defined on the entire T_pM for every $p \in M$ by the Hopf-Rinow theorem [24, p.34].

Example 2.7 (Riemannian manifold as a metric space). *Let M be a connected compact C^k Riemannian manifold. For any $x, y \in M$, there is a C^1 path connecting x and y , that is, a C^1 map $\gamma : [0, 1] \rightarrow M$ with $\gamma(0) = x$ and $\gamma(1) = y$. We can define the length of γ by*

$$L(\gamma) = \int_0^1 \left\| \frac{d\gamma}{dt} \right\| dt.$$

Now we have

$$d(x, y) = \inf \{ L(\gamma) : \gamma \text{ is a } C^1 \text{ path connecting } x \text{ and } y \},$$

which makes M a metric space.

Moreover, M is a complete metric space by the Hopf-Rinow theorem.

Example 2.8 (The space of C^1 maps). *Let M, N be C^1 Riemannian manifolds, $C^1(M, N)$ be the set of all C^1 maps from M to N .*

Recall that the Riemannian metric on M induces a metric on the tangent bundle TM naturally.

Define a metric dist_1 on $C^1(M, N)$ as follow:

$$\text{dist}_1(f, g) = \text{dist}_0(df, dg)$$

for $f, g \in C^1(M, N)$.

2.3 Topological Dynamics

In this subsection, X denotes a locally compact separable metric space, and f is always continuous.

Definition 2.9 (Topological dynamics). *A topological space X with a continuous map $f : X \rightarrow X$ is called a discrete-time dynamical system.*

In this thesis, every dynamical system means the discrete-time dynamical system.

Definition 2.10 (Periodic points). *A point $x \in X$ is called periodic point of $f : X \rightarrow X$ with period $n \in \mathbb{N}$, if $f^n(x) = x$. The set of all periodic points of $f : X \rightarrow X$ is denoted by $\text{Per}(f)$. The smallest positive $n \in \mathbb{N}$, such that $f^n(x) = x$ is called minimal period of x .*

Definition 2.11 (Non-wondering points). *A point $x \in X$ is called a non-wondering point of $f : X \rightarrow X$, if for any neighborhood U of x , there exists $n \in \mathbb{N}$ such that $f^n(U) \cap U$ is a non-empty set. The set of all non-wondering points of $f : X \rightarrow X$ is denoted by $NW(f)$.*

Definition 2.12 (Topological transitivity). *A topological dynamical system $f : X \rightarrow X$ is called topologically transitive if for any two non-empty open sets U and V , $\exists N \in \mathbb{Z}$ such that $f^N(U) \cap V \neq \emptyset$.*

Definition 2.13 (Topological mixing). *A topological dynamical system $f : X \rightarrow X$ is called topologically mixing if for any two non-empty open sets U and V , $\exists N > 0$ such that $f^n(U) \cap V \neq \emptyset$ for all $n > N$.*

From the definitions above, it is obvious that if a topological dynamical system is topologically mixing then it is topologically transitive. Usually, there are topologically transitive maps which are not topologically mixing, such as an irrational rotation of the circle. However, for Anosov diffeomorphisms, topological mixing is equivalent to topological transitivity, which would be shown in Section 4.2.

Definition 2.14 (Topological conjugacy). *A map $f : M \rightarrow M$ is called topologically conjugate to a map $g : N \rightarrow N$ if there exist a homeomorphism $h : M \rightarrow N$ such that $f = h^{-1}gh$.*

3 Hyperbolic Sets and Anosov Diffeomorphisms

In this section, we denote M a C^1 manifold, U a non-empty open subset of M , $f : U \rightarrow f(U) \subset M$ a C^1 diffeomorphism, and $df : TM \rightarrow TM$ the differential of f . Several proofs require too much materials beyond the preliminaries we write before; therefore, for these proofs, we only give a sketch of proof or provide the references.

Definition 3.1 (Hyperbolic sets). *A compact, f -invariant subset $\Lambda \subset M$ is called hyperbolic if there are C^1 Riemannian metric, $\lambda \in (0, 1)$, $C > 0$ such that*

1. $T_\Lambda M = E^s \oplus E^u$
2. $df_x E^s(x) = E^s(f(x))$ and $df_x E^u(x) = E^u(f(x))$
3. $\|df_x^n v\| \leq C\lambda^n \|v\|$ for all $v \in E^s(x)$ and $n > 0$
4. $\|df_x^{-n} v\| \leq C\lambda^n \|v\|$ for all $v \in E^u(x)$ and $n > 0$

Proposition 3.2. *$E^s(x)$ and $E^u(x)$ continuously depend on x .*

Proof. We will prove for any converged sequence $\{x_n\}_{n \in \mathbb{N}}$ in Λ with the limit x we have $\lim_{n \rightarrow \infty} E^s(x_n) = E^s(x)$ and $\lim_{n \rightarrow \infty} E^u(x_n) = E^u(x)$.

Assuming that $\dim E^s(x_i) = k$, a constant, we let $w_{1,i}, w_{2,i}, \dots, w_{k,i}$ be an orthonormal basis of $E^s(x_i)$. Let i tend to infinity, then we can get w_1, \dots, w_k , an orthonormal subset with property $\|df_x^n w_j\| \leq C\lambda^n \|w_j\|$ for all $n \in \mathbb{N}$. Therefore, $\{w_1, \dots, w_k\} \subset E^s(x)$, which implies $\lim_{n \rightarrow \infty} E^s(x_n) \subset E^s(x)$ and $\dim E^s(x) \geq k$.

Similarly, we can get $\dim E^u(x) \geq s - k$, where $\dim T_x M = s$.

Besides, by $s = \dim T_x M = \dim E^s(x) + \dim E^u(x) \geq s - k + \dim E^s(x)$, $k \geq \dim E^s(x)$. Therefore, $\dim E^s(x) = k$ and w_1, \dots, w_k form a basis for $E^s(x)$. It means $\lim_{n \rightarrow \infty} E^s(x_n) = E^s(x)$. Similarly, $\lim_{n \rightarrow \infty} E^u(x_n) = E^u(x)$. \square

Definition 3.3 (Anosov diffeomorphisms). *A diffeomorphism $f : M \rightarrow M$ is called Anosov diffeomorphism if M is a hyperbolic set of f .*

We will use the following theorem to prove the structural stability of Anosov diffeomorphisms:

Theorem 3.4 (Shadowing Theorem). *Let Λ be a hyperbolic set of $f : U \rightarrow f(U) \subset M$. Then there is an open set $O \subset U$ containing Λ and $\epsilon_0, \delta_0 > 0$, satisfying:*

$\forall \epsilon > 0, \exists \delta > 0$ such that for any $g : O \rightarrow M$ with $\text{dist}_1(g, f) < \epsilon_0$, any homeomorphism $h : X \rightarrow X$ of a topological space X , and any continuous map $\phi : X \rightarrow O$ with $\text{dist}_0(\phi h, g\phi) < \delta$ there is a continuous map $\psi : X \rightarrow O$ with $\psi h = g\psi$ and $\text{dist}_0(\phi, \psi) < \epsilon$.

Moreover, if $\psi' h = g\psi'$ for some $\psi' : X \rightarrow O$ with $\text{dist}_0(\phi, \psi') < \delta_0$ then $\psi' = \psi$.

Sketch of proof. Refer to [1, p.566] for the details.

We will apply the contraction mapping principle in the proof. Notice that the desired map $\psi : X \rightarrow O$ is a fixed point of

$$F : C(X, O) \rightarrow C(X, M), \psi \rightarrow g\psi h^{-1}.$$

In order to keep $\text{dist}_0(\phi, \psi) < \epsilon$, for sufficiently small $\theta > 0$, consider the map

$$\mathcal{A} : B_\theta(\phi) \rightarrow C_\phi(X, TM),$$

$$\text{given by } (\mathcal{A}\psi)(y) = \exp_{\phi(y)}^{-1} \psi(y),$$

where \exp_p^{-1} is the inverse of exponential map of M at p , $B_\theta(\phi) = \{\psi \in C(X, O) : \text{dist}_0(\phi, \psi) < \theta\}$, $C_\phi(X, TM) = \{v \in C(X, TM) : v(y) \in T_{\phi(y)}M, \forall y \in X\}$.

Then we define $F^\phi = \mathcal{A}F\mathcal{A}^{-1}$, which can be decomposed into linear and nonlinear parts, that is, $F^\phi(v) = dF_0^\phi v + H(v)$. Now we define

$$T(v) = -(dF_0^\phi - \text{Id})^{-1}H(v).$$

T can be proved to be contracting on a sufficiently small neighborhood of ϕ . By contraction mapping principle, we obtain v , the fixed point of T . Now we find a fixed point of F , $\mathcal{A}^{-1}v$. \square

Besides, we introduce a criterion for hyperbolicity.

Firstly, given a continuous direct sum decomposition $T_\Lambda M = E^s \oplus E^u$, that is for every $x \in \Lambda$ and $v \in T_x M$ we have $v = v^s + v^u$ where $v^s \in E^s(x)$ and $v^u \in E^u(x)$, and $E^s(x)$, $E^u(x)$ continuously depend on x .

For $\alpha > 0$, let us define

$$K_\alpha^s(x) = \{v \in T_x M : \|v^u\| \leq \alpha \|v^s\|\},$$

$$K_\alpha^u(x) = \{v \in T_x M : \|v^s\| \leq \alpha \|v^u\|\}.$$

They are called *stable and unstable cones of size α* , respectively.

Let Λ be a hyperbolic set of f .

Proposition 3.5. $\forall \alpha > 0, \exists \epsilon > 0$ such that $\forall x \in \Lambda$, we get $df_x(K_\alpha^u(x)) \subset \{0\} \cup \text{int}(K_\alpha^u(f(x)))$ and $df_{f(x)}^{-1}(K_\alpha^s(f(x))) \subset \{0\} \cup \text{int}(K_\alpha^s(x))$.

Proof. For $x \in \Lambda$, let $v = v^s + v^u \in K_\alpha^u(x)$. We will prove $\|df_x(v^s)\| < \alpha \|df_x(v^u)\|$.

Firstly, note that there exists a metric such that the hyperbolic set Λ is with the constant $C = 1$ and $\lambda \in (0, 1)$, which is called an *adapted metric*. So by the hyperbolicity $\|df_x v^s\| \leq \lambda \|v^s\|$. Then by the definition of $K_\alpha^u(x)$ we have $\|df_x v^s\| \leq \alpha \lambda \|v^u\|$. Applying the hyperbolicity again, $\|df_x v^s\| \leq \alpha \lambda^2 \|df_x v^u\| < \alpha \|df_x v^u\|$.

The second equation can be proved similarly. \square

Proposition 3.6. $\forall \delta > 0, \exists \alpha > 0$ such that $\forall x \in \Lambda$, $\|df_x^{-1}v\| \leq (\lambda + \delta)\|v\|$ for $v \in K_\alpha^u(x)$ and $\|df_x v\| \leq (\lambda + \delta)\|v\|$ for $v \in K_\alpha^s(x)$.

Proof. For $x \in \Lambda$, let $v = v^s + v^u \in K_\alpha^s(x)$. Choose the adapted metric as the proposition above.

Notice that when $\|v\| = 1$, $\|df_x v^u\| \rightarrow 0$ as $\alpha \rightarrow 0$, by continuity. So for every $\delta > 0$, there exists an α such that $\|df_x v^u\| \leq \delta$.

Then we have $\|df_x v\| \leq \|df_x v^s\| + \|df_x v^u\| \leq \lambda \|v\| + \delta \|v\| = (\lambda + \delta)\|v\|$.

The first inequality for $v \in K_\alpha^u(x)$ can be proved similarly. \square

Proposition 3.7. Let $f : U \rightarrow f(U) \subset M$ be a C^1 diffeomorphism and Λ be a compact invariant set for $f : U \rightarrow M$. If there are $\alpha > 0, \lambda \in (0, 1)$ and subspaces decomposition $T_\Lambda M = \tilde{E}^s \oplus \tilde{E}^u$ such that $\tilde{E}^s(x)$ and $\tilde{E}^u(x)$ continuously depend on x , and the stable cones K_α^s and unstable cones K_α^u of size α determined by the decomposition satisfy the following properties for any $x \in \Lambda$:

1. $df_x K_\alpha^u(x) \subset K_\alpha^u(f(x))$ and $df_{f(x)}^{-1} K_\alpha^s(f(x)) \subset K_\alpha^s(x)$,
2. $\|df_x v\| < \|v\|$ for $0 \neq v \in K_\alpha^s(x)$ and $\|df_x^{-1}v\| < \|v\|$ for $0 \neq v \in K_\alpha^u(x)$.

Then Λ is a hyperbolic set for f .

Proof. Firstly by the compactness we have $\exists \lambda \in (0, 1)$ such that $\|df_x v\| \leq \lambda \|v\|$ for $v \in K_\alpha^s(x)$, and $\|df_x^{-1} v\| \leq \lambda \|v\|$ for $v \in K_\alpha^u(x)$. For $x \in \Lambda$, we define

$$E^s(x) = \bigcap_{n \geq 0} df_{f^n(x)}^{-n} K^s(f^n(x)),$$

$$E^u(x) = \bigcap_{n \geq 0} df_{f^{-n}(x)}^n K^u(f^{-n}(x)).$$

Then Λ is hyperbolic set of f with decomposition $T_\Lambda M = E^s \oplus E^u$, constant λ and $C = 1$. \square

At the end of the section, we present an important concept, stable and unstable manifolds. Their denseness is highly relevant to the topological transitivity of Anosov diffeomorphisms, which will be shown in Section 4.

Definition 3.8 (Stable and unstable manifolds). *For a hyperbolic set Λ of $f : U \rightarrow f(U) \subset M$ and every $x \in \Lambda$, we define stable manifolds of x by*

$$W^s(x) = \{y \in M : d(f^n(x), f^n(y)) \rightarrow 0 \text{ as } n \rightarrow \infty\}$$

and unstable manifolds of x by

$$W^u(x) = \{y \in M : d(f^{-n}(x), f^{-n}(y)) \rightarrow 0 \text{ as } n \rightarrow \infty\}.$$

Definition 3.9. *For an Anosov diffeomorphism $f : M \rightarrow M$ and any $\epsilon > 0$, we define*

$$W_\epsilon^s(x) = \{y \in M : d(f^n(x), f^n(y)) < \epsilon, \forall n \in \mathbb{N}_0\},$$

$$W_\epsilon^u(x) = \{y \in M : d(f^{-n}(x), f^{-n}(y)) < \epsilon, \forall n \in \mathbb{N}_0\}.$$

The stable and unstable manifolds theorem[2, p.121] have shown the existence of W^u and W^s . In this thesis, we do not need all of the results of the theorem. Therefore, a part of results are collected in the following proposition.

We denote the distances along the stable and the unstable manifolds by d^s and d^u respectively.

Proposition 3.10. *For an Anosov diffeomorphism $f : M \rightarrow M$, there are $\lambda \in (0, 1)$, $C_p > 0$, $\epsilon, \delta > 0$, and a splitting $T_x M = E^s(x) \oplus E^u(x)$ for all $x \in M$ such that:*

1. $df_x(E^s(x)) = E^s(f(x))$, and $df_x(E^u(x)) = E^u(f(x))$;
2. $\forall v^s \in E^s(x)$, $\|df_x v^s\| \leq \lambda \|v^s\|$, and $\forall v^u \in E^u(x)$, $\|df_x^{-1} v^u\| \leq \lambda \|v^u\|$;
3. $\forall y \in W^s(x)$, $d^s(f(x), f(y)) \leq d^s(x, y)$, and $\forall y \in W^u(x)$, $d^s(f^{-1}(x), f^{-1}(y)) \leq d^u(x, y)$;
4. $f(W^s(x)) = W^s(f(x))$, and $f(W^u(x)) = W^u(f(x))$;
5. $T_x W^s(x) = E^s(x)$, and $T_x W^u(x) = E^u(x)$;
6. if $d(x, y) < \delta$, $W_\epsilon^s(x) \cap W_\epsilon^u(x) = \{p_{x,y}\}$; moreover, $p_{x,y}$ continuously depends on x, y , and $d^s(p_{x,y}, x) \leq C_p d(x, y)$, $d^u(p_{x,y}, y) \leq C_p d(x, y)$.

Proof. It is a direct result from Hadamard-Perron theorem in [1, p.242] and Proposition 5.9.1 in [2, p.128]. \square

4 Properties of Anosov Diffeomorphisms

4.1 Anosov diffeomorphisms are structurally stable

Definition 4.1 (Structurally stability). *A C^1 map $f : M \rightarrow M$ is called structurally stable if there exists a neighborhood U of f in the C^1 topology such that every $g \in U$ is topologically conjugate to f .*

Lemma 4.2. *Let Λ be a hyperbolic set of $f : U \rightarrow M$. There is an open set $U(\Lambda)$ containing Λ and $\epsilon_0 > 0$ such that if $K \subset U(\Lambda)$ is a compact invariant subset of a diffeomorphism $g : U \rightarrow M$ with $\text{dist}_1(g, f) < \epsilon_0$, then K is a hyperbolic set of g .*

Proof. Notice that $E^s(x)$ and $E^u(x)$ continuously depends on x by Proposition 3.2; therefore, we can continuously extend the subspace decomposition to $T_{U(\Lambda)}M = E^s \oplus E^u$. By Proposition 3.5 and Proposition 3.6, the cones $K_\alpha^s(x)$ and $K_\alpha^u(x)$ with $x \in U(\Lambda)$ determined by the decomposition have the following properties:

1. $df_x K_\alpha^u(x) \subset K_\alpha^u(f(x))$ and $df_{f(x)}^{-1} K_\alpha^s(f(x)) \subset K_\alpha^s(x)$;
2. $\|df_x v\| \leq (\lambda + \delta)\|v\|$ for $0 \neq v \in K_\alpha^s(x)$ and $\|df_x^{-1} v\| \leq (\lambda + \delta)\|v\|$ for $0 \neq v \in K_\alpha^u(x)$.

And we have $\text{dist}_1(f, g) = \text{dist}_0(df, dg) < \epsilon_0$. For a sufficiently small ϵ_0 we can replace f by g .

That is, for $0 \neq v \in K_\alpha^s(x)$, $\|dg_x v\| \leq \text{dist}_0(df, dg)\|v\| + (\lambda + \delta)\|df_x v\| < \|v\|$. Similarly, $\|dg_x v\| < \|v\|$ for $0 \neq v \in K_\alpha^u(x)$, $dg_x K_\alpha^u(x) \subset K_\alpha^u(g(x))$ and $dg_{g(x)}^{-1} K_\alpha^s(g(x)) \subset K_\alpha^s(x)$.

Then by Proposition 3.7, the proof is finished if we let K be an compact invariant subset of $U(\Lambda)$. \square

Corollary 4.3. *Anosov diffeomorphisms form an open set in $\text{Diff}^1(M)$.*

Proof. Let $f : M \rightarrow M$ be an Anosov diffeomorphism. By Lemma 4.2, there is $\epsilon_f > 0$ such that for any diffeomorphism $g : M \rightarrow M$ with $\text{dist}_1(g, f) < \epsilon_f$, g is an Anosov diffeomorphism. \square

Theorem 4.4. *Let $\Lambda \subset M$ be a hyperbolic set of the diffeomorphism $f : U \rightarrow M$. Then for every open set $V \subset U$ containing Λ and every $\epsilon > 0$, there exists $\delta > 0$ such that if $g : V \rightarrow M$ with $\text{dist}_1(f, g) < \delta$, there is a hyperbolic set K of g and a homeomorphism $\chi : K \rightarrow \Lambda$ such that $\chi g = f \chi$ and $\text{dist}_0(\chi, \text{Id}) < \epsilon$.*

Proof. By the Shadowing Theorem, for every $\epsilon > 0$ there exists $\delta > 0$ such that if we are given $X = \Lambda$, $h = f|_\Lambda$, $\phi = \text{Id}_\Lambda$ and $g : V \rightarrow M$ is chosen to satisfy $\text{dist}_1(f, g) < \delta$, there is a continuous map $\psi : \Lambda \rightarrow U$ with $\psi f|_\Lambda = g\psi$.

Set $K = \psi(\Lambda)$. By Lemma 4.2, there exist $\epsilon_0 > 0$ such that K is a hyperbolic set of g if $\text{dist}_1(f, g) < \epsilon_0$. Therefore, we can let δ above be not greater than ϵ_0 to keep the hyperbolicity of K .

Now since K is hyperbolic set of g we can apply the Shadowing theorem again by taking $X = K$, $h = g|_K$, $\phi = \text{Id}_K$. There is a continuous map $\psi' : K \rightarrow U$ with $\psi' g|_K = f|_\Lambda \psi'$.

By the equation above, we can get $\psi' \psi f|_\Lambda = f|_\Lambda \psi' \psi$ and $g|_K \psi \psi' = \psi \psi' g|_K$. By the uniqueness, we get $\psi \psi' = \text{Id}_\Lambda$ and $\psi \psi' = \text{Id}_K$. That is $\psi^{-1} = \psi'$.

Finally, let the homeomorphism $\chi : K \rightarrow \Lambda$ be ψ' to finish the proof. \square

Corollary 4.5. *Anosov diffeomorphisms are structurally stable.*

Proof. Let $f : M \rightarrow M$ be an Anosov diffeomorphism. By Theorem 4.3, for every $\epsilon > 0$, there is $\delta > 0$ such that for any $g : M \rightarrow M$ with $\text{dist}_1(f, g) < \delta$, we get a $h : M \rightarrow M$ with $\text{dist}_0(h, \text{Id}) < \epsilon$ and $hg = fh$. Therefore, f is structurally stable by Definition 4.1. \square

4.2 Topological transitivity of Anosov diffeomorphisms

In this subsection, M is a connected compact Hausdorff smooth Riemannian manifold.

Lemma 4.6. *Let $f : M \rightarrow M$ be an Anosov diffeomorphism. Then $\text{Per}(f)$ is dense in $\text{NW}(f)$.*

Proof. It is sufficient to prove that for any $x \in \text{NW}(f)$ and $\epsilon > 0$ there exists $p \in \text{Per}(f)$ such that $d(x, p) < 2\epsilon$.

For every $\epsilon > 0$, choose $\delta \in (0, \epsilon)$ by the Shadowing theorem. Because $V = \{z \in M : d(z, x) < \delta/2\}$ is a neighborhood of $x \in \text{NW}(f)$, there is a $n \in \mathbb{N}$ such that $f^n(V) \cap V$ is nonempty. Take a $z \in V \cap f^{-n}(V)$.

Let $h : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ by $h(m) = m+1$, $\phi : \mathbb{Z}_n \rightarrow M$ by $\phi(m) = z_m$ where $z_m = f^m(z)$. Since $d(z_{m+1}, f(z_m)) = 0$ for every $m < n-1$ in \mathbb{Z}_n and $d(z_n, f(z_{n-1})) \leq d(z, x) + d(x, f^n(z)) \leq \delta$, by the Shadowing theorem, there exists $\{p_m\}_{m \in \mathbb{Z}_n}$ such that $p_{m+1} = f(p_m)$, $p_0 = f(p_{n-1})$ and $d(p_m, z_m) < \epsilon$.

We get $d(p_0, x) \leq d(p_0, z) + d(z, x) < 2\epsilon$. \square

Definition 4.7 (ϵ -dense). *A subset $A \subset X$ is called to be ϵ -dense in a metric space (X, d) if $d(x, A) < \epsilon$ for every $x \in X$.*

Theorem 4.8. *Let $f : M \rightarrow M$ be an Anosov diffeomorphism. Then the following are equivalent:*

1. $\text{NW}(f) = M$,
2. Every unstable manifold is dense in M ,
3. Every stable manifold is dense in M ,

4. f is topologically transitive,

5. f is topologically mixing.

Proof. $1 \Rightarrow 2$: We need to prove that $\forall \epsilon > 0, \forall x \in M$, the unstable manifold $W^u(x)$ is ϵ -dense in M .

Firstly, for any $\epsilon > 0$, we construct a $\epsilon/4$ -dense set with finite elements in $\text{Per}(f)$. By Lemma 4.6, $\text{Per}(f)$ is dense in M , which implies $M = \bigcup_{x \in \text{Per}(f)} B(x, \epsilon/4)$. Therefore, by compactness of M , there are N elements in $\text{Per}(f)$, $\{x_i\}_{i=1,2,\dots,N}$, such that $M = \bigcup_{i=1}^N B(x_i, \epsilon/4)$. That is, $\forall x \in M, \exists i$ such that $\text{dist}(x, x_i) < \epsilon/2$ and $x_i \neq x$.

Let the product of their periods be P and define $g = f^P$. We prove that g and f have the same unstable manifolds, so we can use the unstable manifolds of g to instead of those of f . Let $\widetilde{W}^u(x) = \{y \in M : d(g^n(x), g^n(y)) \rightarrow 0 \text{ as } n \rightarrow \infty\}$ be the unstable manifolds of g at x . Obviously, $W^u(x) \subset \widetilde{W}^u(x)$. Because f is C^1 , for every $n \in \mathbb{N}$, $d(f^{nP+1}(x), f^{nP+1}(y)) < d(f^{nP}(x), f^{nP}(y))$. Then $\widetilde{W}^u(x) \subset W^u(x)$.

And we need to notice the following lemma:

Lemma 4.9. $\exists q \in \mathbb{N}$ such that if for some y, x_i, x_j such that $\text{dist}(W^u(y), x_i) < \epsilon/2$ and $\text{dist}(x_i, x_j) < \epsilon/2$, then we have $\text{dist}(g^{nq}(W^u(y)), x_i) < \epsilon/2$ and $\text{dist}(g^{nq}(W^u(y)), x_j) < \epsilon/2$.

Proof. We choose appropriate ϵ to make sure that $W^u(y) \cap W_e^s(x_i) \neq \emptyset$ for a sufficiently small e by Proposition 3.10.

Then let $z \in W^u(y) \cap W_e^s(x_i)$. Because $z \in W^s(x_i) = \bigcup_{n=0}^{\infty} f^{-n}W_e^s(f^n(x_i))$, there is a t_0 such that $\text{dist}(g^t(z), x_i) < \epsilon/2$ for all $t \geq t_0$ (We need note that $g^t(x_i) = x_i$). Then we have $\text{dist}(g^t(W^u(z)), x_j) < \epsilon$ by the triangle inequality.

We can gain a $w \in W^u(g^t(z)) \cap W_{e'}^s(x_j)$ for a sufficiently small e' by Proposition 3.10, for the distance between $g^t(W^u(z))$ and x_j is small enough. Because $w \in W^s(x_j)$, there is a s_0 such that $\text{dist}(g^\tau(w), x_j) < \epsilon/2$ for all $\tau \geq s_0$.

Finally, we make $q = s_0 + t_0$ to finish the lemma. \square

Since the set $\{x_i\}_{i=1,2,\dots,N}$ is $\epsilon/4$ -dense, $\forall y \in M, \exists x_s$ such that $\text{dist}(g^{nq}(W^u(y)), x_s) < \epsilon/2$. And for any x_t there is a chain in $\{x_i\}_{i=1,2,\dots,N}$ which connects x_t and x_s and the distance between two consecutive points less than $\epsilon/2$, because M is compact and connected. Note that the length of the chain will not be larger than N .

Therefore, for every $z \in M$, if we choose i such that $\text{dist}(z, x_i) < \epsilon/2$ and let $y = g^{-Nq}(x)$ for any x , we can obtain $\text{dist}((W^u(x)), z) \leq \text{dist}((W^u(x)), x_i) + \text{dist}(x_i, z) < \epsilon$. It means $W^u(x)$ is ϵ -dense in M for any $x \in M$ and $\epsilon > 0$.

$1 \Rightarrow 3$: Similarly.

$2 \Rightarrow 5$: We need to prove that $\exists N, \forall n > N, f^n(U) \cap V \neq \emptyset$ for any non-empty sets U and V , by the definition of topologically mixing.

Let us choose $x, y \in M$ and $\delta > 0$ such that $W_\delta^u(x) \subset U$ and $B(y, \delta) \subset V$. Notice that $f^n(W_\delta^u(x)) \subset f^n(U)$, for any $n \in \mathbb{N}$.

Lemma 4.10. If for every $x \in M, W^u(x)$ is dense in M , then $\forall \delta > 0, \exists R = R(\delta) > 0$ such that every ball of radius R in every unstable manifold is ϵ -dense in M .

Proof. Because $W^u(x) = \bigcup_R W_R^u(x)$ is dense in M , there is R which depends on x and $W_R^u(x)$ is $\epsilon/2$ -dense in M . Since W^u is a continuous foliation, we get $\delta(x)$ such that $W_R^u(y)$ is ϵ -dense, $\forall y \in B(x, \delta)$.

Now we have $M = \bigcup_{x \in M} B(x, \delta(x))$. By compactness we find a finite collection for those balls $B(x, \delta(x))$. Then choose the maximal $R(x)$ for those balls to finish the lemma. \square

Notice that $\exists N, \forall n > N, W_R^u(f^n(x)) \subset f^n(W_\delta^u(x))$.

By the lemma above, for $n > N, B(y, \delta) \cap W_R^u(f^n(x)) \neq \emptyset$, which means $V \cap f^n(U) \neq \emptyset$. Hence f is topologically mixing.

$3 \Rightarrow 5$: Similarly.

$5 \Rightarrow 4 \Rightarrow 1$: It is obvious by their definitions. \square

5 Examples

In this section, we firstly introduce a procedure to construct an Anosov diffeomorphism on nilmanifolds. Then we will follow the procedure to give several examples of Anosov diffeomorphisms.

5.1 A brief introduction to nilmanifolds

Before giving the detail of the construction, we need to give a brief introduction to nilmanifolds.

Definition 5.1 (Lie groups). *A group G with a smooth manifold structure is called Lie group, if the following group operations are smooth:*

$$\begin{aligned}(a, b) &\rightarrow ab, \\ a &\rightarrow a^{-1},\end{aligned}$$

where $a, b \in G$.

Definition 5.2 (Lie group homomorphisms). *Given Lie group G and H , A map $g : G \rightarrow H$ is called a Lie group homomorphism if it is a smooth map and also a group homomorphism from G to H .*

It is called a Lie group isomorphism if it is also a diffeomorphism.

A Lie group isomorphism $g : G \rightarrow G$ is called a Lie group automorphism.

And in the rest of the subsection, we use G to represent a Lie group.

By the definition above, we can define the following smooth map, left transition for every $g \in G$:

$$L_g(h) = gh.$$

In fact, $L_g : G \rightarrow G$ is a diffeomorphism with smooth inverse $L_{g^{-1}}$. Therefore, for an arbitrary smooth vector field X , we can obtain $(L_g)_*X$, a smooth vector field, which is defined as following:

$$((L_g)_*X)_h = (dL_g)_{g^{-1}h}(X_{g^{-1}h}),$$

where $h \in G$.

Now we give a brief introduction to Lie algebras.

Definition 5.3 (Lie algebra). *Let L be a vector space. A map from $L \times L$ to L denoted $(X, Y) \rightarrow [X, Y]$ is called a Lie bracket on L if*

1. *the map is bilinear;*
2. $\forall X \in L, [X, X] = 0$;
3. $\forall X, Y, Z \in L, [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

The last condition is called Jacobi identity.

The vector space L with a Lie bracket is called a Lie algebra.

Definition 5.4 (Lie algebra homomorphisms). *A vector space homomorphism $\phi : L \rightarrow L'$ is called a Lie algebra homomorphism, if $\phi([X, Y]) = [\phi(X), \phi(Y)]$.*

A vector space isomorphism $\phi : L \rightarrow L'$ is called a Lie algebra isomorphism if $\phi([X, Y]) = [\phi(X), \phi(Y)]$.

A Lie algebra isomorphism $\phi : L \rightarrow L$ is called a Lie algebra automorphism.

We call a Lie algebra automorphism is hyperbolic, if its eigenvalues are away from the unit circle.

Definition 5.5 (Structure constants of Lie algebras). *Let L be a Lie algebra with a basis X_1, X_2, \dots, X_n . We have*

$$[X_i, X_j] = \sum_{k=1}^n a_{ij}^k X_k$$

for any $i, j \leq n$. We call those a_{ij}^k are structure constants of L .

Example 5.6 (Lie algebra of G). *Now we define a Lie algebra associated with Lie group G . Firstly we let*

$$\text{Lie}(G) = \{X \text{ is a smooth vector field on } G : (L_g)_*X = X, \forall g \in G\}.$$

And for any smooth vector fields X, Y on G , define a smooth vector field $[X, Y]$ by

$$[X, Y]f = XYf - YXf,$$

for every smooth map $f : G \rightarrow G$. (Note that $[X, Y]$ is a smooth vector field although we have not shown that; refer to this [15].)

$\text{Lie}(G)$ is called Lie algebra of G , and the map $(X, Y) \rightarrow [X, Y]$ is a Lie bracket on $\text{Lie}(G)$.

We define $[x, y] = x^{-1}y^{-1}xy$. Notice the lower central series of G :

$$\begin{aligned}\gamma_1(G) &= G, \\ \gamma_2(G) &= [\gamma_1(G), G], \\ &\dots \\ \gamma_{k+1}(G) &= [\gamma_k(G), G].\end{aligned}$$

Definition 5.7 (Nilpotent Lie group). G is called a nilpotent Lie group, if $\exists n \in \mathbb{N}$ such that $\gamma_n(G) = 1$.

Definition 5.8 (c -step nilpotent Lie group). G is called a c -step nilpotent Lie group, if $\gamma_c(G) \neq 1$ and $\gamma_{c+1}(G) = 1$.

Definition 5.9 (Lattice). N is a simply-connected and nilpotent Lie group. Let Γ be a discrete subgroup of G . It is called a lattice in G if G/Γ is a compact quotient space.

Note that the quotient space G/Γ must be a smooth manifold by Theorem 21.29 in Lee's book [15]. Some nilpotent Lie groups do not admit any lattices. Therefore, we need the following criterion:

Proposition 5.10 (Maltsev's criterion). A nilpotent Lie group N admits some lattices if and only if all of the structure constants of $\text{Lie}(N)$ are in \mathbb{Q} .

Proof. Refer to Theorem 2.12 in Raghunathan's book [16]. □

Definition 5.11 (Nilmanifolds). A differential manifold N/Γ is called a nilmanifold, if N is a simply-connected nilpotent Lie group and Γ is a lattice in N .

Example 5.12 (\mathbb{R}^n as a Lie group). \mathbb{R}^n is a Lie group with the group operation $+$. Obviously, \mathbb{R}^n is an abelian group, so a nilpotent Lie group.

Moreover, it is well-known that \mathbb{R}^n is simply-connected.

Example 5.13 (\mathbb{T}^n as a Lie group). It is easy to verify that \mathbb{Z}^n is a normal subgroup of \mathbb{R}^n . And it is also a closed Lie subgroup of \mathbb{R}^n , for it is discrete subgroup. Therefore, $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is a Lie group.

And $\mathbb{T}^n \cong \mathbb{S}^1 \times \mathbb{S}^1 \times \dots \times \mathbb{S}^1$, which implies compactness. So \mathbb{Z}^n is a lattice of \mathbb{R}^n .

5.2 Example: An Anosov diffeomorphism on tori

Now we can introduce the procedure to construct an Anosov diffeomorphism on a nilmanifold. Let N be a simply-connected nilpotent Lie group, Γ be a lattice in N , $f : N \rightarrow N$ be a Lie group automorphism of N such that $f(\Gamma) = \Gamma$, and $df_{\text{Id}} : T_{\text{Id}}N \rightarrow T_{\text{Id}}N$, the Lie algebra automorphism induced by f , is hyperbolic. Obviously, the procedure induces a Anosov diffeomorphism $\tilde{f} : N/\Gamma \rightarrow N/\Gamma$ on a nilmanifold.

The procedure is reasonable because Smale pointed out that if a Lie algebra admits a hyperbolic Lie algebra automorphism then it must be nilpotent, which forces N to be nilpotent and Γ to be a uniform discrete subgroup [3].

Then we will construct an Anosov diffeomorphisms on \mathbb{T}^2 as an example.

Example 5.14. As we show in Example 5.12 and Example 5.13, \mathbb{R}^2 is a simply-connected nilpotent Lie group and \mathbb{Z}^2 is a lattice in \mathbb{R}^2 .

Firstly, we define a linear map $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Because $\det L = 1$, we have $L(\mathbb{Z}^2) = \mathbb{Z}^2$. And we denote the eigenvalues of L by λ and $1/\lambda$, where the value of λ is $\frac{3-\sqrt{5}}{2}$.

The map L induces a diffeomorphism $F_L : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ by

$$F_L(x, y) = (2x + y, x + y) \pmod{1},$$

where $x, y \in \mathbb{R}/\mathbb{Z}$.

Now we prove that F_L is an Anosov diffeomorphism.

Proposition 5.15. F_L is an Anosov diffeomorphism.

Proof. For any $p \in \mathbb{T}^2$, the differential of F_L at p is a vector space isomorphism $(dF_L)_p : T_p \mathbb{T}^2 \rightarrow T_{F_L(p)} \mathbb{T}^2$. The matrix of $(dF_L)_p$ in term of the coordinate basis is $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

Then we have a direct sum decomposition, $T_p \mathbb{T}^2 = E_p^s \oplus E_p^u$, where E_p^s, E_p^u are eigenspaces associated with $\lambda, 1/\lambda$, respectively. It is easy to verify that $\{E_p^s\}_{p \in \mathbb{T}^2}$ ($\{E_p^u\}_{p \in \mathbb{T}^2}$) form the stable (unstable) distribution of F_L . Therefore, F_L is an Anosov diffeomorphism. \square

Finally, we prove that F_L is topologically mixing. It is sufficient to prove $\text{Per}(F_L)$ is dense in \mathbb{T}^2 . By Lemma 4.6 and Theorem 4.8, F_L is topologically mixing.

Proposition 5.16. *The periodic points of F_L are dense in \mathbb{T}^2 .*

Proof. We claim that all points with a rational coordinate are periodic points. The claim implies the proposition is true. Now we prove the claim. Firstly, we let $p = (s/q, t/q) \in \mathbb{T}^2$, an arbitrary point with a rational coordinate.

Consider the set of rational points on \mathbb{T}^2 with denominator q . It is a finite set with q^2 element and contains $\{F_L^n(s/q, t/q)\}_{n \geq 0}$, which means that $\exists M, N \in \mathbb{N}$ such that $F_L^M(s/q, t/q) = F_L^N(s/q, t/q)$.

Now recall that F_L is a diffeomorphism. The proof is finished. \square

5.3 Example: An Anosov diffeomorphism on a nontoral manifold

In this subsection, we firstly introduce the Heisenberg group and use it to construct another example of Anosov diffeomorphism.

We define the Heisenberg group

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}$$

with the matrix multiplication.

The Lie algebra of H is given by

$$\text{Lie}(H) = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}$$

with generators

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now we can begin our construction. We let $G = H \times H = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \middle| A, B \in H \right\}$, which is a simply-connected nilpotent Lie group. Moreover, the basis of $\text{Lie}(G)$ contains

$$X_1 = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}, Y_1 = \begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix}, Z_1 = \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix},$$

$$X_2 = \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix}, Y_2 = \begin{pmatrix} 0 & 0 \\ 0 & Y \end{pmatrix}, Z_2 = \begin{pmatrix} 0 & 0 \\ 0 & Z \end{pmatrix}.$$

For any $\lambda > 1$, we define $F : \text{Lie}(G) \rightarrow \text{Lie}(G)$ given by

$$\begin{aligned} F(X_1) &= \lambda X_1, & F(X_2) &= \lambda^{-1} X_2, \\ F(Y_1) &= \lambda^2 Y_1, & F(Y_2) &= \lambda^{-2} Y_2, \\ F(Z_1) &= \lambda^3 Z_1, & F(Z_2) &= \lambda^{-3} Z_2, \end{aligned}$$

which is a hyperbolic Lie algebra automorphism on $\text{Lie}(G)$.

By Proposition 5.10, G admits some lattices because $[X, Y] = Z$ and other Lie brackets of generators are zero. Actually we can give a lattice Γ defined by $\exp_{\text{Id}}(\gamma)$ with

$$\gamma = \left\{ \begin{pmatrix} A & 0 \\ 0 & \sigma(A) \end{pmatrix} \in \text{Lie}(G) \middle| A = \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \text{ for } x, y, z \text{ are algebraic integer in } \mathbb{K} \right\},$$

where $\mathbb{K} = \mathbb{Q}(\sqrt{3}) = \{a + b\sqrt{3} : a, b \in \mathbb{Q}\}$, $\sigma : a + b\sqrt{3} \rightarrow a - b\sqrt{3}$.

Notice we have the following decomposition:

$$\text{Lie}(G) = \text{Lie}(G^u) \oplus \text{Lie}(G^s),$$

where $G^u = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \middle| A \in \text{Lie}(H) \right\}$, and $G^s = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \middle| B \in \text{Lie}(H) \right\}$.

By Lie group and Lie algebra theory, there is a unique automorphism $f : G \rightarrow G$ with $df|_{\text{Id}} = F$ and $f(\Gamma) = \Gamma$. It induces an Anosov diffeomorphism of G/Γ .

References

- [1] A. Katok, B. Hasselblatt. Introduction to the modern theory of dynamical systems. Cambridge University Press, 1995
- [2] M. Brin, S. Garrett. Introduction to dynamical systems. Cambridge University Press, 2002
- [3] S. Smale. Differentiable dynamical systems. Bull. Amer. Math. Soc. 73, 1967. no. 6. 747-817.
- [4] D. V. Anosov. Geodesic flows on compact manifolds of negative curvature, Trudy mat. Inst, 1967. V.A Steklova 90
- [5] A. Manning. There are no new Anosov diffeomorphisms on tori. Amer. J. Math. 96, 1974. 422-429.
- [6] J. Franks. Anosov diffeomorphisms. Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968) p. 6193 Amer. Math. Soc., Providence, R.I., 1970
- [7] G. A. Margulis. Problems and conjectures in rigidity theory. Mathematics: frontiers and perspectives, 161174, Amer. Math. Soc., Providence, RI, 2000
- [8] A. Avez. Topologie des diffeomorphismes d'Anosov qui possdent un invariant integral. C.R. Acad. Sei, 1967
- [9] S. Newhouse. On codimension one Anosov diffeomorphisms. Amer. J. Math. 92, 1970. 761-770.
- [10] Y. Benoist and F. Labourie. Sur les diffeomorphismes Anosov affines feuilletages stable et instable differentiables. Invent. Math. 111, 1993. no. 2, 285-308.
- [11] L. Flaminio and A. Katok. Rigidity of symplectic Anosov diffeomorphisms on lowdimensional tori. Ergodic Theory Dynam. Systems 11, 1991. no. 3, 427-441.
- [12] E. Ghys. Holomorphic Anosov systems. Invent. Math. 119, 1995. 585-614.
- [13] B. Kalinin and V. Sadovskaya. On local and global rigidity of quasi-conformal Anosov diffeomorphisms. J. Inst. Math, 2003. Jussieu 2, no. 4, 567-582.
- [14] F. T. Farrell and L. E. Jones. Anosov diffeomorphisms constructed from $\pi_1(\text{Diff}(\text{Sn}))$. Topology 17, 1978. no. 3, 273-282.
- [15] J. M. Lee. Introduction to Smooth Manifolds. 2nd Edition. Springer. 2013
- [16] M. S. Raghunathan. Discrete subgroups of Lie groups. Ergebnisse de Mathematik und ihrer Grenzgebiete, 1972. Springer-Verlag. MR 0507234
- [17] A. I. Mal'cev. On a class of homogeneous spaces, AMS Translation no. 39, 1951
- [18] Jorge Lauret , Cynthia , E. Will. Nilmanifolds of dimension ≤ 8 admitting Anosov diffeomorphisms. Transactions of the American Mathematical Society, 2008

- [19] Wim Malfait. Anosov Diffeomorphisms on Nilmanifolds of Dimension at Most Six. *Geometriae Dedicata* 79(3):291-298, 2000
- [20] Jorge Lauret. Examples of Anosov diffeomorphisms. *Journal of Algebra* 262(1):201-209, 2003
- [21] Karel Dekimpe, Kelly Verheyen. Anosov diffeomorphisms on nilmanifolds modelled on a free nilpotent Lie group. *Dynamical Systems* 24(1):117-121 March 2009
- [22] Hugh L. Porteous. Anosov diffeomorphisms of flat manifolds. *Topology* Vol. 11, pp. 307-315. Pergamon Press, 1972. Printed in Great Britain, 1971
- [23] Karel Dekimpe, Jonas Der. Existence of Anosov diffeomorphisms on infra-nilmanifolds modeled on free nilpotent Lie groups. *Topol. Methods Nonlinear Anal.* Volume 46, Number 1 (2015), 165-189.
- [24] Jost. *Riemannian Geometry and Geometric Analysis* 5th Edition.