Notes on Actuarial Statistics

September 2, 2019

Outline

Actuarial Notations

Notations:

- (x) or x: a life aged x
- T_x: the future lifetime of x
- $F_x(t)$: the distribution of T_x; the probability of dying at age $x+t$
- $S_x(t)$: the probability of surviving at age $x+t$

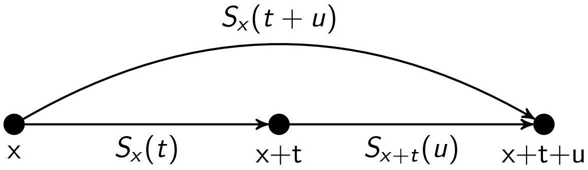
Relation:

$$S_x(t) = 1 - F_x(t)$$

Result 2.4

The probability that (x) survives to at least $x + t + u$ is equal to the probability of surviving to $x + t$ multiplied by the probability of $x + t$ surviving to $x + t + u$:

$$S_x(t + u) = S_x(t) \cdot S_{x+t}(u)$$



Actuarial notations:

- ${}_tq_x$: $F_x(t)$; the probability of dying at age $x+t$
- ${}_tp_x$: $S_x(t)$; the probability of surviving at age $x+t$
- "x dies within t years, given that x has survived u years":

$$ {}_{u|t}q_x = Pr[u < T_x \leq u+t] = S_x(u) - S_x(u+t) $$

Formula

- $$ {}_tp_x = 1 - {}_tq_x $$
- $$ {}_tp_x \cdot {}_up_{x+t} = {}_{t+u}p_x $$
  $$ {}_xp_0 \cdot {}_tp_x = {}_{t+x}p_0 $$

Example

Given $S_x(t) = e^{-t}$. Find ${}_tp_y$ and ${}_{t|u}q_y$:

- ${}_tp_x = e^{-t}$
- ${}_tp_y =_{u+t} p_x / {}_up_x = e^{-t}$
- ${}_{t|u}q_x = S_x(t) - S_x(u+t) = {}_tp_x - {}_{u+t}p_x = e^{-t} - e^{-(t+u)}$

Example

Given ${}_1p_x = 0.99$ and ${}_1p_{x+1} = 0.9$, find ${}_2p_x$.

$${}_2p_x = {}_{1+1}p_x = {}_1p_x \cdot {}_1p_{x+1} = 0.99 \cdot 0.9$$

The instantaneous rate of decrement due to death μ_x is defined as

$$\mu_x = \lim_{dx \to x} \frac{1}{dx} Pr[T_0 \leq x + dx | T_0 > x].$$

Result 2.9 and Result 2.18

Re-write it using $S_0(x)$:

$$\mu_x == \frac{-d/dx\ S_0(x)}{S_0(x)}$$

Let f_0 be the probability density function of T_0:

$$\mu_x = \frac{f_0(x)}{S_0(x)}$$

General case:

$$\mu_{x+t} == \frac{f_x(t)}{S_x(t)}$$

where $F_x(t) = {}_tq_x = \int_0^t f_x(s)ds$ (PDF of T_x).

Remark:

- Given $S_x(t)$, find μ_{x+t}:

$$\mu_{x+t} = \frac{-d/dt\,(S_x(t))}{S_x(t)} = -d/dt \ln(S_x(t))$$

- Given μ_{x+t}, find $S_x(t)$:

$$S_x(t) = \exp\left(\int_0^t (-\mu_{x+s})ds\right)$$

Example

Given $S_x(t) = (10-t)^2/100$, $0 \le t < 10$, find μ_{x+t}:

$$\mu_{x+t} = -\frac{-2(10-t)}{(10-t)^2} = \frac{2}{10-t}$$

Given $\mu_{x+t} = \frac{2}{10-t}$, find $S_x(t)$:

$$S_x(t) = \exp\left(\int_0^t (-\frac{2}{10-s})ds\right)$$

In actuarial notation:

Result 2.20

$$ {}_t q_x = \int_0^t {}_s p_x \mu_{x+s} ds $$

Mean of T_x:

- $\mathring{e}_x$: the complete expectation of life; $\mathrm{E}\,T_x$.

$$\mathring{e}_x = \int_0^\infty t f_x(t)dt = \int_0^\infty {}_tp_x dt$$

- $\mathring{e}_{x:\overline{n}|}$:

$$\mathring{e}_{x:\overline{n}|} = \int_0^n {}_tp_x dt$$

- Relation:

$$\mathring{e}_x = \mathring{e}_{x:\overline{n}|} + {}_np_x \mathring{e}_{x+n}.$$

Example

Given $l_x = (100-x)^{0.5}$ for $0 \le x \le 100$ and $\mathring{e}_{36:\overline{28}|} = 24.67$. Calculate

$$\int_0^{28} t \cdot {}_tp_{36} \cdot \mu_{36+t} dt.$$

- Simplify $\int_0^{28} t \cdot {}_tp_{36} \cdot \mu_{36+t} dt$:

$$\begin{aligned}\int_0^{28} t \cdot {}_tp_{36} \cdot \mu_{36+t} dt &= \int_0^{28} t \cdot {}_tp_{36} \cdot \frac{-{}_tp_{36}{}'}{{}_tp_{36}} dt \\ &= -\int_0^{28} t \cdot {}_tp_{36}{}' dt \\ &= -\left[28 \cdot {}_{28}p_{36} - \int_0^{28} {}_tp_{36} dt\right]\end{aligned}$$

Example

Given $l_x = (100 - x)^{0.5}$ for $0 \leq x \leq 100$ and $\mathring{e}_{36:\overline{28}|} = 24.67$. Calculate

$$\int_0^{28} t \cdot {}_tp_{36} \cdot \mu_{36+t} dt.$$

- Simplify $\int_0^{28} t \cdot {}_tp_{36} \cdot \mu_{36+t} dt$:

$$\int_0^{28} t \cdot {}_tp_{36} \cdot \mu_{36+t} dt = -\left[28 \cdot {}_{28}p_{36} - \int_0^{28} {}_tp_{36} dt\right]$$

- ${}_{28}p_{36} = \frac{l_{36+28}}{l_{36}} = \frac{3}{4}$; $\int_0^{28} {}_tp_{36} dt = \mathring{e}_{36:\overline{28}|} = 24.67$.
- $\int_0^{28} t \cdot {}_tp_{36} \cdot \mu_{36+t} dt = 3.67$

Example

Show that

$$e_x \le \mathring{e}_x \le \mathring{e}_{x+1} + 1.$$

First, we prove $\mathring{e}_x \le \mathring{e}_{x+1} + 1$:

$$\begin{aligned}
\mathring{e}_x &= \int_0^\infty {}_tp_x dt \\
&= \int_0^1 {}_tp_x dt + \int_1^\infty {}_tp_x dt \\
({}_tp_x \le 1) \quad &\le 1 + \int_1^\infty {}_tp_x dt \\
&= 1 + \int_1^\infty p_x \cdot {}_{t-1}p_{x+1} dt
\end{aligned}$$

Example

(continue...) Show that

$$e_x \leq \mathring{e}_x \leq \mathring{e}_{x+1} + 1.$$

First, we prove $\mathring{e}_x \leq \mathring{e}_{x+1} + 1$:

$$\begin{aligned} \mathring{e}_x &= 1 + \int_1^\infty p_x \cdot {}_{t-1}p_{x+1} dt \\ (p_x \leq 1) \quad &\leq 1 + \int_1^\infty {}_{t-1}p_{x+1} dt \\ (u = t-1) \quad &= 1 + \int_0^\infty {}_u p_{x+1} du \\ &= 1 + \mathring{e}_{x+1} \end{aligned}$$

Curtate Future Lifetime

- The integer part of T_x
$$K_x = \lfloor T_x \rfloor$$
e.g. $\lfloor 1.999 \rfloor = 1$.
- $e_x := \mathrm{E} K_x$.
Note:
$$e_x = \mathrm{E} K_x = \sum_{k=0}^{\infty} k \cdot Pr(K_x = k)$$
$$= \sum_{k=0}^{\infty} k \cdot Pr\left(T_x \in [k, k+1)\right)$$
$$= \sum_{k=0}^{\infty} k \cdot \left({}_k p_x - {}_{k+1} p_x\right)$$
$$= \sum_{k=1}^{\infty} {}_k p_x$$

Example

(continue...) Show that

$$e_x \le \dot{e}_x \le \dot{e}_{x+1} + 1.$$

Note:

- $e_x = \sum_{k=1}^{\infty} {}_k p_x = p_1 + 2p_2 + 3p_3 + \dots$
- $\dot{e}_x = \int_0^{\infty} {}_t p_x dt = \int_0^1 {}_t p_x dt + \int_1^2 {}_t p_x dt + \int_2^3 {}_t p_x dt + \dots$
- ${}_s p_x$ is decreasing in s.

Outline

Notations:

- l_x: number alive at age x

Remark

$$l_{x+t}/l_x = {}_tp_x.$$

- **uniform distribution of deaths** (UDD)

$$ {}_sq_x = sq_x $$

- **constant force of mortality** (CFM)

$$ {}_sp_{x+t} = (p_x)^s $$

Example

Standard Ultimate Life Table, "LTAM tables" in GauchoSpace.

- Find l_{40}.
  ($= 99,338.3$. Directly find it in SULT)
- Compute ${}_{10}p_{30}$.
  ($= l_{30+10}/l_{30} = 0.9966$. Use the formula above)
- Compute ${}_{1}q_{35}$.
  (Directly find it in SULT; or $q_{35} = 1 - p_{35} = 1 - \frac{l_{36}}{l_{35}} = 0.000391$)
- Or explain it: the probability of being dead in the next 1 year.
  How many people die in the next 1 year?

$$l_{35} - l_{36}$$

Main Problem:

Now, we know how to compute ${}_{10}p_{30}$. But how to compute

$${}_{0.75}p_{30.5}?$$

- **uniform distribution of deaths** (UDD)

$${}_{s}q_x = sq_x$$

where $0 \leq s \leq 1$.

Useful Formula

Under UDD,

$$ {}_s q_{x+t} = \frac{s q_x}{1 - t q_x} $$

where $(s + t) \leq 1$.

Example

We CANNOT directly use

$$ {}_{0.75}p_{30.5} = {}_{0.75}p_{30+0.5} $$

because $0.75 + 0.5 > 1$.

Useful Formula

- **UDD**.

$$ {}_s q_x = s q_x $$

where $0 \le s \le 1$.

- $$ {}_t p_x \cdot {}_u p_{x+t} = {}_{t+u} p_x $$

Example

Compute ${}_{0.75}p_{30.5}$: (**Hint**: $30.5 = 30 + 0.5$; $x + t$)

$$ {}_{0.75}p_{30.5} = \frac{{}_{0.5}p_{30} \cdot {}_{0.75}p_{30.5}}{{}_{0.5}p_{30}} $$
$$ = \frac{{}_{1.25}p_{30}}{{}_{0.5}p_{30}} = \frac{p_{30} \cdot {}_{0.25}p_{31}}{{}_{0.5}p_{30}} $$
$$ = $$

Note: Last equality. p and q.

- **constant force of mortality** (CFM)

$$ {}_{s}p_{x+t} = (p_x)^s $$

where $s + t < 1$.

Example

(EXAMPLE 5 and EXAMPLE 8) Calculate

$$ {}_{0.4}q_{40.2} $$

- **Under CFM**:

$$ {}_{0.4}q_{40.2} = 1 - {}_{0.4}p_{40.2} = 1 - p_{40}^{0.4} = 0.000211 $$

- **Under UDD** $(0.4 + 0.2 \leq 1)$:

$$ {}_{0.4}q_{40.2} = \frac{0.4q_{40}}{1 - 0.2q_{40}} = 0.000211 $$

- **constant force of mortality** (CFM)

$$_{s}p_{x+t} = (p_x)^s$$

where $s + t < 1$.

Example

(EXAMPLE 9) Calculate

$$_{0.7}q_{70.6}$$

- The following method is **WRONG**

$$_{0.7}q_{70.6} = 1 - (p_{70})^{0.7}$$

because $0.7 + 0.6 > 1$.

Useful Results

- **constant force of mortality** (CFM)

$$ {}_s p_{x+t} = (p_x)^s $$

where $s + t < 1$.

-

$$ {}_t p_x \cdot {}_u p_{x+t} = {}_{t+u} p_x $$

Example

(EXAMPLE 9) Calculate ${}_{0.7}q_{70.6}$: $(= 1 - {}_{0.7}p_{70.6})$

$$ \begin{aligned} {}_{0.7}p_{70.6} &= \frac{{}_{0.6}p_{70} \cdot {}_{0.7}p_{70.6}}{{}_{0.6}p_{70}} \\ &= \frac{{}_{1.3}p_{70}}{{}_{0.6}p_{70}} = \frac{p_{70} \cdot {}_{0.3}p_{71}}{{}_{0.6}p_{70}} \end{aligned} $$

Notations:

- ${}_tq_{[x]+s}$: Pr[a life currently aged $x+s$ who was select at age x survives to age $x+s+t$]
- ${}_tp_{[x]+s} := 1 - {}_tq_{[x]+s}$.
- **Note**:
  - ${}_tq_{[x]+s}$ depends on t, $[x]$, s;
  - ${}_tq_{x+s}$ only depends on t, $x+s$.

Select & Ultimate Model

Example (from textbook)

- **Background**: Men who need to undergo surgery because they are suffering from a particular disease. The surgery is complicated, so only 50% of them could survive for a year. And if they do survive for a year, they are fully cured.
- **Select**: time for 1st surgery
- **Question**: the probability that a man aged 60 who is just about to have surgery will be alive at age 70.

Select & Ultimate Model

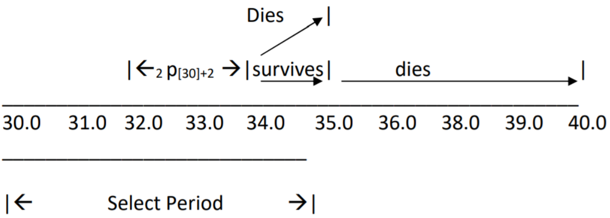
Example (from textbook)

- **Background**: Men who need to undergo surgery because they are suffering from a particular disease.
- The surgery is complicated, so only 50% of them could survive for **a year**. And if they do survive for a year, they are fully cured.
  Select period: 1 year.
- **Select**: time for 1st surgery
- **Question**: the probability that a man aged 60 who is just about to have surgery will be alive at age 70. ${}_{10}p_{[60]}$
- **Solution**:
  $= Pr[\text{live 1 year after surgery}] \times Pr[\text{live 9 year from age 61}]$
  $= 0.5 \times {}_{9}p_{61} = 0.5 \times \frac{l_{70}}{l_{60}}$

Example (Lecture notes: EXAMPLE 13 (textbook 3.10))

Represent ${}_{2|6}q_{[30]+2}$ using $l_{[x]+t}$ or l_{x+t}. Select period 5 years.

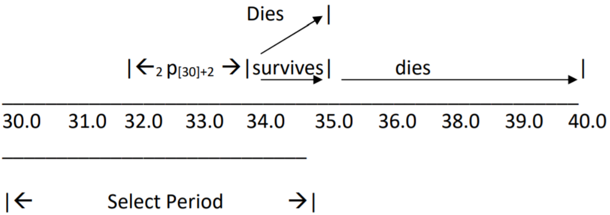
- ${}_{2|6}q_{[30]+2}$: The probability that a life now aged 32 who was select 2 years ago will die between 34 and 40.
- die between 34 and 40 = (die between 34 and 35) or (survive between 34 and 35; then die between 35 and 40)
- die between 34 and 40 = not survive between 34 and 40



Example (Lecture notes: EXAMPLE 13 (textbook 3.10))

Represent ${}_{2|6}q_{[30]+2}$ using $l_{[x]}$ or l_x. Select period 5 years.

- ${}_{2|6}q_{[30]+2}$: The probability that a life now aged 32 who was select 2 years ago will die between 34 and 40.
- $$\begin{aligned} {}_{2|6}q_{[30]+2} &= {}_{2}q_{[30]+2} \cdot {}_{6}q_{[30]+4} \\ &= \frac{l_{[30]+4}}{l_{[30]+2}} \cdot \left(q_{[30]+4} + p_{[30]+4} \cdot {}_{5}q_{[30]+5}\right) \end{aligned}$$



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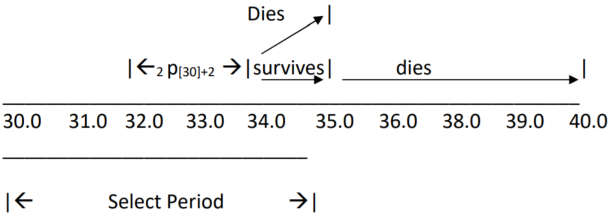
- $$\begin{aligned} {}_{2|6}q_{[30]+2} &= {}_{2}q_{[30]+2} \cdot {}_{6}q_{[30]+4} \\ &= \frac{l_{[30]+4}}{l_{[30]+2}} \cdot \left(1 - \frac{l_{40}}{l_{[30]+4}}\right) \end{aligned}$$



Outline

Interest notations:

- Assume we fix
  i: interest rate for 1 year; put 1\$ in the bank, get $(1+i)$\$ after 1 year.
- **Related concept**
  - $i^{(12)}/12$: "interest rate for 1 mouth";
    put 1\$ in the bank, get $1+i^{(12)}/12$ after 1 month. For 1 year, get
    $(1+i^{(12)}/12)^{12} = 1+i$
    Nominal rate, compounded p times per year $i^{(p)}$.
  - **Force of interest**. "interest rate for a very small time interval"
    Let $p \to \infty$:
    $$\lim_{p\to\infty}(1+\frac{i^{(p)}}{p})^p = e^{\lim_{p\to\infty} i^{(p)}} = 1+i$$
    Denote $\delta = \lim_{p\to\infty} i^{(p)}$. $1+i=e^{\delta}$.

- If I want to have 1$ at time t, how much money I should put it into bank at time 0?

$$e^{-\delta t}$$

- Now T_x is a random variable. At time T_x, I need to have 1$. At present (time 0), the 1$ worth

$$\mathbb{E}e^{-\delta T_x}$$

Notations:

- **Expected present value**

$$\bar{A}_x := \mathbb{E}(v^{T_x}) = \mathbb{E}(e^{-\delta T_x}) = \int_0^\infty e^{-\delta t} {}_t p_x \mu_{x+t} dt.$$

- $Z = e^{-\delta T_x} = v^{T_x}$

- $$A_x = \mathbb{E}[v^{K_x+1}] = vq_x + v^2{}_{1|}q_x + v^3{}_{2|}q_x + \dots.$$

Reminder: $K_x := \lfloor T_x \rfloor$; ${}_{k|}q_x = Pr[K_x = k] = Pr[k \leq T_x < k+1]$.

- $Z = v^{K_x+1}$. (We don't need δ anymore)

Example (Compute variance of Z)

Useful formula:

$$\mathrm{V}[Z] = \mathbb{E}[Z^2] - (\mathbb{E}[Z])^2$$

And

$$\begin{aligned} \mathbb{E}[Z^2] &= \mathbb{E}[(v^2)^{T_x}] \\ &= \mathbb{E}[e^{-2\delta T_x}] \\ &= \int_0^\infty e^{-2\delta t} {}_t p_x \mu_{x+t} dt \end{aligned}$$

We write ${}^2\bar{A}_x = \mathbb{E}[Z^2]$. Then

$$\mathrm{V}[Z] = {}^2\bar{A}_x - (\bar{A}_x)^2.$$

Example (Compute $P(Z \leq 0.5)$)

$$\begin{aligned} Pr[Z \leq 0.5] &= Pr[e^{-\delta T_x} \leq 0.5] \\ &= Pr[T_x > \log(2)/\delta] \\ &= {}_u p_x \end{aligned}$$

where $u = \log(2)/\delta$.

Notations:

- **(continuous) n-year term insurance**

$$\bar{A}^{1}_{x:\bar{n}|} := \int_0^n e^{-\delta t} {}_tp_x \mu_{x+t} dt$$

- **(discrete) n-year term insurance**

$$A^{1}_{x:\bar{n}|} := \sum_{k=0}^{n-1} v^{k+1} {}_{k|}q_x$$

- Reminder Whole life insurance:

$$\bar{A}_x := E[v^{T_x}] = \int_0^\infty e^{-\delta t} {}_tp_x \mu_{x+t} dt$$

$$A_x := E[v^{K_x+1}] = \sum_{k=0}^{\infty} v^{k+1} {}_{k|}q_x$$

- (n-term insurance) Present value of \$1:

$$Z = \begin{cases} v^{T_x} & T_x \leq n \\ 0 & \text{o.w.} \end{cases}$$

$$Z = \begin{cases} v^{K_x+1} & K_x \leq n-1 \\ 0 & \text{o.w.} \end{cases}$$

Example (Compute the variance of Z)

For a 2-year term insurance on (x), calculate Var[Z] (given benefit $1).
First, compute $E[Z]$:

$$\begin{aligned} E[Z] &= A^{1}_{x:\bar{2}|} \\ &= vq_x + v^2{}_1|q_x \end{aligned}$$

where q can be computed using life table and $v = \frac{1}{i+1}$.
And compute $E[Z^2]$:

$$E[Z^2] = v^2 q_x + v^4{}_1|q_x$$

Then use $Var[Z] = E[Z^2] - (E[Z])^2$.

Pure Endowment:

- Present value of $ 1:

$$Z = \begin{cases} 0 & T_x < n \\ v^n & T_x \geq n \end{cases}$$

- Definition

$$_nE_x := E[Z] = v^n {}_np_x$$

Example

For a Pure Endowment written on a life age (x), compute Var[Z].

- $$E[Z] = v^n {}_n p_x$$

- $$E[Z^2] = v^{2n} {}_n p_x$$

- $$\begin{aligned} Var[Z] &= E[Z^2] - (E[Z])^2 \\ &= v^{2n} {}_n p_x - v^{2n} {}_n p_x^2 \\ &= v^{2n} ({}_n p_x)({}_n q_x) \end{aligned}$$

Endowment:

- Present value of \$ 1:

$$Z = \begin{cases} v^{T_x} & T_x < n \\ v^n & T_x \geq n \end{cases}$$

- Definition

$$\bar{A}_{x:\bar{n}|} := E[Z] = \bar{A}^1_{x:\bar{n}} + {}_nE_x.$$

- Discrete case

$$A_{x:\bar{n}|} := A^1_{x:\bar{n}} + {}_nE_x.$$

Deferred insurance benefits:

- $$Z = \begin{cases} 0 & T_x \notin [u, u+n) \\ e^{-\delta T_x} & T_x \in [u, u+n) \end{cases}$$

- Definition:

$$_{u|}\bar{A}^{1}_{x:\bar{n}|} = E[Z] = \int_u^{u+n} e^{-\delta t}\, {}_t p_x \mu_{x+t} dt.$$

- $$_{u|}\bar{A}^{1}_{x:\bar{n}|} = \bar{A}^{1}_{x:\overline{u+n}|} - \bar{A}^{1}_{x:\bar{u}|}$$

Summary

The annual case

Notation	Z	$E[Z]$
A_x	$Z = v^{K_x+1}$	$vq_x + v^2{}_1\vert q_x + \dots$
$A^1_{x:\bar{n}\vert}$	$Z = v^{K_x+1} \cdot 1\{K_x \leq n-1\}$	$\sum_{k=0}^{n-1} v^{k+1}{}_k\vert q_x$
$A_{x:\bar{n}\vert}^{\ \ 1}$	$Z = v^n \cdot 1\{T_x \geq n\}$	$v^n{}_n p_x$
$A_{x:\bar{n}\vert}$	$Z = v^{\min(K_x+1,n)}$	$A^1_{x:\bar{n}\vert} + v^n{}_n p_x$

Insurance notes, page 5.

Approximation:

- $$\bar{A}_x \approx \frac{i}{\delta} A_x$$

  (under UDD, it is "=")

- $$\bar{A}_{x:\bar{n}|} \approx \frac{i}{\delta} A^{1}_{x:\bar{n}|} + v^n {}_n p_x$$

Approximation:

- $$\bar{A}_x \approx (1+i)^{1/2} A_x$$

  (the claim acceleration approach)

- $$\bar{A}_{x:\bar{n}|} \approx (1+i)^{1/2} A^{1}_{x:\bar{n}|} + v^n {}_n p_x$$

Table in textbook:

x	$\bar{A}_x / A_x$
20	1.0246
40	1.0246
60	1.0246
80	1.0248
100	1.0261
120	1.0368

Note $i = 5\%$. $i/\delta = 1.0248$ and $(1+i)^{1/2} = 1.0247$.

Continue: **the claim acceleration approach**
Annual case: pay at the end of year
Monthly case: pay at the end of month (m=12)

...

- $A_x^{(m)} = v^{1/m}{}_{1/m}q_x + v^{2/m}{}_{1/m|1/m}q_x + \cdots = \sum_{k=0}^{\infty} v^{k+\frac{1}{m}}{}_{\frac{k}{m}|\frac{1}{m}}q_x$
- Re-write it in annual case: $\frac{m+1}{2m}$ is the average time of payment.

$$A_x^{(m)} \approx q_x v^{\frac{m+1}{2m}} + {}_1|q_x v^{1+\frac{m+1}{2m}} + \ldots$$

- Take $v^{\frac{m-1}{2m}}$ out:

$$A_x^{(m)} \approx (1+i)^{\frac{m-1}{2m}} \cdot A_x$$

Whole life annuity-due

- **Reminder**:
  If I have 1\$, how much money I will have after n years?
  Answer: v^n\$.
- Whole life annuity-due

$$\ddot{a}_x = 1 + v \cdot p_x + v^2 \cdot {}_2p_x + v^3 \cdot {}_3p_x + \dots$$

$$\bar{a}_x = \int_0^\infty e^{-\delta t} {}_tp_x dt$$

- Y, the present value random variable (for whole life annuity-due, discrete case). Then

$$E[Y] = E[I(T_x > 0)] + v \cdot E[I(T_x > 1)] + v^2 \cdot E[I(T_x > 2)] + \dots$$

Whole life annuity-due

- Y, the present value random variable.

$$E[Y] = E[I(T_x > 0)] + v \cdot E[I(T_x > 1)] + v^2 \cdot E[I(T_x > 2)] + \ldots$$

- $I(T_x > 0) = I(0 < T_x < 1) + I(1 \leq T_x < 2) + I(2 \leq T_x < 3) + \ldots$. Then

$$E[I(T_x > 0)] = Pr[K_x = 0] + Pr[K_x = 1] + Pr[K_x = 2] + \ldots$$
$$= \sum_{k=0}^{\infty} {}_{k|}q_x$$

Whole life annuity-due

- Y, the present value random variable.

$$E[Y] = E[I(T_x > 0)] + v \cdot E[I(T_x > 1)] + v^2 \cdot E[I(T_x > 2)] + \dots$$

-

$$E[I(T_x > 0)] = \sum_{k=0}^{\infty} {}_{k|}q_x$$

$$E[I(T_x > 1)] = \sum_{k=1}^{\infty} {}_{k|}q_x$$

$$\vdots$$

Whole life annuity-due

- Y, the present value random variable.

$$\begin{aligned} E[Y] &= E[I(T_x > 0)] + v \cdot E[I(T_x > 1)] + v^2 \cdot E[I(T_x > 2)] + \dots \\ &= {}_0|q_x + {}_1|q_x \cdot (1 + v) + {}_2|q_x \cdot (1 + v + v^2) + \dots \\ &= \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|}\, {}_k|q_x \end{aligned}$$

where $\ddot{a}_{\overline{k+1}|} = \sum_{t=0}^{k} v^t$ is a series of annuities-certain.
(Example 5.1, pg 111)

Whole life annuity-due

- Let $Y = \frac{1-v^{T_x}}{\delta}$ be the present value random variable.

$$\bar{a}_x = \frac{1 - \bar{A}_x}{\delta}$$

- It also holds for discrete case:

$$\ddot{a}_x = \frac{1 - A_x}{d}$$

- Or term annuities due

$$\ddot{a}_{x:\bar{n}|} = \frac{1 - A_{x:\bar{n}|}}{d}$$

Compute its variance (continuous case):

$$Var[Y] = \frac{1}{\delta^2}({}^2\bar{A}_x - (\bar{A}_x)^2) = \frac{2}{\delta^2}(\bar{a}_x - {}^2\bar{a}_x) - (\bar{a}_x)^2.$$

where ${}^2\bar{a}_x = \int_{t=0}^{\infty} e^{-2\delta t}\,{}_tp_x dt$

Discrete case:

$$Var[Y] = \frac{2}{d}[\ddot{a}_x - {}^2\ddot{a}_x] + {}^2\ddot{a}_x$$

where ${}^2\ddot{a}_x = \sum v^{2k}\,{}_kp_x$

Whole life annuity-due

Example

Given $\delta = 0.05$ and $(\mu) = 0.02$. Compute $\bar{a}_{x:\bar{10}|}$ and $Var[Y]$.

Solution.

- Directly compute it by definition.

$$\bar{a}_{x:\bar{10}|} = \int_0^{10} e^{-0.05t} {}_tp_x dt$$

($_tp_x$ can be computed using $\mu = 0.02$.)

- For $Var[Y]$,

$$Var[Y] = \frac{2}{\delta^2}(\bar{a}_{x:\overline{10}|} - {}^2\bar{a}_{x:\overline{10}|}) - (\bar{a}_{x:\overline{10}|})^2$$

Whole life annuity-due

Example

Given: $\delta = 0.05$; Mortality is uniformly distributed throughout life (DeMoivre) with $\omega = 100$. Compute $\bar{a}_{35}$ and $Var[Y]$.

Solution.

- We can directly compute ${}^2\bar{A}_{35}$ and $\bar{A}_{35}$ using DeMoivre Law.
- ${}^2\bar{A}_{35} = \frac{1}{100-35}\,{}^2\bar{a}_{\overline{100-35}|}$ and
- $\bar{A}_{35} = \frac{1}{100-35}\bar{a}_{\overline{100-35}|}$
  (annuities certain, $\int_0^{65} e^{-\delta t} dt$)

Annuities payable (m) times per year

- Annuities payable (m) times due

$$\ddot{a}_x^{(m)} = \frac{1 - A_x^{(m)}}{d^{(m)}}$$

$d^{(m)}$: the nominal rate of discount compounded m times.
$= p(1 - v^{1/m})$.

- Applying the UDD

$$\ddot{a}_x^{(m)} = \alpha(m)\ddot{a}_x - \beta(m)$$

- Woolhouse approximation:

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\delta + \mu_x)$$

Example

Mortality follows the ILT. UDD assumption. $i = 0.06$.
Calculate $\ddot{a}^{(4)}_{25:\overline{20}|}$.

- UDD: $\ddot{a}^{(4)}_{25:\overline{20}|} = \alpha(4)\ddot{a}^{(4)}_{25:\overline{20}|} - \beta(4)\cdot(1 - {}_{20}E_{25})$
- By ILT table: ${}_{20}E_{25} = 0.29873$.
- $\alpha(4) = 1.00027$, and $\beta(4) = 0.38424$.

$$\alpha(4) = \frac{id}{i^{(4)}d^{(4)}}; \quad \beta(4) = \frac{i - i^{(m)}}{i^{(m)}d^{(m)}}$$

where $i^{(m)} = m\left[(1+i)^{1/m} - 1\right]$ and $d^m = m(1-(1+i)^{-1/m})$.

Example

You are given that δ(force of interest) and μ (force of mortality) are each constant and that $\bar{a}_x = 12.50$. Use the Woolhouse approximation to 3 terms to find $\ddot{a}_x^{(12)}$.

Woolhouse approximation:

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\delta + \mu)$$

And $\bar{a}_x = \frac{1-\bar{A}_x}{\delta} = \frac{1}{\mu+\delta}$. Then we can solve $\delta + \mu$. $\ddot{a}_x = \frac{1}{1-e^{-(\delta+\mu)}}$.

Deferred Annuities

- u-year deferred annuity-due

$$ {}_{u|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\bar{u}|} = \sum_{k=0}^{\infty} v^{u+k}\, {}_{u+k}p_x $$

- $$ {}_{u|}\ddot{a}_x = v^u\, {}_{u}p_x \sum_{k=0}^{\infty} v^k\, {}_{k}p_{x+u} = {}_{u}E_x \ddot{a}_{x+u} $$

Deferred Annuities

Example (Deferred n-term annuity immediate)

The force of mortality follows Makeham's law with $A = 0.0002$, $B = 0.0000003$ and $c = 1.10000$. The annual effective rate of interest is 5%. Calculate ${}_{1|}a_{70:\bar{2}|}$.

- Makeham's law of mortality: $\mu_x = A + Bc^x$.
  $\implies \ {}_tp_x = \exp\left(-At - \frac{Bc^x}{\ln c}(c^t - 1)\right)$.
- We want to compute one-year deferred two-year annuity immediate.

$$ {}_{1|}a_{70:\bar{2}|} = v^2 {}_2p_{70} + v^3 {}_3p_{70}. $$

- Compute it using given numbers (≈ 1.75819).

Solve ${}_tp_x$

Following Result:

$$ {}_tp_x = \exp\left(-\int_0^t \mu_{x+u} du\right) $$

μ_x is given. Solve ${}_tp_x$.

Example (Deferred whole life annuity due)

For a 5-year deferred whole life annuity-due of 1 on (x) you are given:

1. $\mu_{x+t} = 0.01$ for $t \geq 1$.
2. $i = 0.04$.
3. $\ddot{a}_{x:\overline{5}|} = 4.542$.

Let S be the sum of annuity payments. Calculate $Pr[S > {}_{5|}\ddot{a}_x]$

- Recall that ${}_{5|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{5}|}$. given in (3)
- Use (1) and CMF to solve $\ddot{a}_x$.

$$\ddot{a}_x = 1 + vp_x\ddot{a}_{x+1} = 1 + vp_x\ddot{a}_x.$$

$$\implies \ddot{a}_x = 1/(1 - \tfrac{1}{1.04} \cdot e^{-0.01}).$$

- $Pr[S > {}_{5|}\ddot{a}_x] = Pr[S > 16.2788]$. We need to compute the probability that (x) survive to year 21. $(= \exp(-0.01 \times 21) = 0.81)$

Guaranteed Annuities

- n-year guaranteed annuity-due

$$\ddot{a}_{\overline{x:\bar{n}|}} = \ddot{a}_{\bar{n}|} + {}_nE_x \ddot{a}_{x+n}$$

- Notice ${}_u|\ddot{a}_x = {}_uE_x \ddot{a}_{x+u}$; so

$$\ddot{a}_{\overline{x:\bar{n}|}} = \ddot{a}_{\bar{n}|} + \ddot{a}_x - \ddot{a}_{x:\bar{n}|}$$

Guaranteed Annuities

Example

At interest rate $i = 0.78$. You are given

1. $\ddot{a}_x = 5.6$
2. $\ddot{a}_{\overline{x:\bar{2}|}} = 5.6459$.
3. $e_x = 8.83$ (complete expectation of life)

Calculate e_{x+1}.

- $\ddot{a}_{\overline{x:\bar{2}|}} = \ddot{a}_{\bar{2}|} + \ddot{a}_x - \ddot{a}_{x:\bar{2}|}$,
  where $\ddot{a}_{\bar{2}|} = 1 + v$ and $\ddot{a}_{x:\bar{2}|} = 1 + vp_x$.
- And $e_x = \sum_{t=1}^{\infty} {}_tp_x = p_x(1 + e_{x+1})$. $({}_{t+1}p_x = p_x \cdot {}_tp_{x+1})$
- Solve e_{x+1}. (≈ 8.29)

- Increasing annuities due.

$$(I\ddot{a})_x = \sum_{t=0}^{\infty} v^t (t+1)\,{}_tp_x.$$

- Geometrically increasing. annuity due

$$\ddot{a}_{x:\overline{n}|i^*} = \sum_{t=0}^{n-1} v^t (1+j)^t\,{}_tp_x.$$

- other...

Outline

Premium

Formula: (**fully-discrete**, level benefit, level premium policies)

- Whole Life with premiums for life

$$P_x = A_x / \ddot{a}_x$$

- n-year term insurance with premiums for n years

$$P^{1}_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} / \ddot{a}_x$$

- n-year Pure endowment with premiums for n years:

$$P_{x:\overset{1}{\overline{n}|}} = A_{x:\overset{1}{\overline{n}|}} / \ddot{a}_x$$

Premium

Formula (continue):

- n-year Endowment insurance with premiums for n years:

$$P_{x:\overline{n}|} = A_{x:\overline{n}|} / \ddot{a}_{x:\overline{n}|}$$

- k-payment Whole life policy:

$${}_kP_x = A_x / \ddot{a}_{x:\overline{k}|}$$

Premium

Plan	Premium	Benefit
Fully discrete	At the start of each year $\ddot{a}_x$	At the end of the year of death (if death benefit) (A_x)
Fully continuous	Continuously $(\bar{a}_x)$	Moment of Death $(\bar{A}_x)$
Semi-continuous	At the start of each year $(\ddot{a}_x)$	Moment of Death $(\bar{A}_x)$

(for whole life)

Premium

Example

Using the Illustrative Table at 6%, find the level annual benefit premium for a 25-year **endowment** insurance issued to (40) with death benefit 1000 and endowment benefit 2000 in **fully discrete** cases.

Solution: use formula (a little bit different)

$$P_{40:\overline{25}} = \left[1000A^1_{40:\overline{25}|} + 2000A_{40:\frac{1}{\overline{25}|}}\right] / \ddot{a}_{40:\overline{25}|}$$

Next step: we need to solve $A^1_{40:\overline{25}|}$, $A_{40:\frac{1}{\overline{25}|}}$ and $\ddot{a}_{40:\overline{25}|}$.

Example

Using the Illustrative Table at 6%, find the level annual benefit premium for a 25-year **endowment** insurance issued to (40) with death benefit 1000 and endowment benefit 2000 in **fully continuous** cases.

Solution: use formula

$$P_{40:\overline{25}} = \left[1000\bar{A}^{1}_{40:\overline{25}|} + 2000A_{40:\overset{\ 1}{\overline{25}|}}\right] / \bar{a}_{40:\overline{25}|}$$

Premium

Example

Under UDD assumption and
(i) $i = 0.04$, (ii) ${}_nE_x = 0.6$, and (iii) $\bar{A}_{x:\bar{n}|} = 0.804$
Calculate $P(\bar{A}_{x:\bar{n}|})$.

Solution: use formula

$$P = \frac{\bar{A}_{x:\bar{n}|}}{\ddot{a}_{x:\bar{n}|}} = 0.155.$$

$\ddot{a}_{x:\bar{n}|}$ is unknown. Use UDD: $A^1_{x:\bar{n}|} = \frac{\delta}{i}\bar{A}^1_{x:\bar{n}|} = 0.200.$

And $A_{x:\bar{n}|} = A^1_{x:\bar{n}|} + {}_nE_x = 0.80.$ Then

$$\ddot{a}_{x:\bar{n}|} = \frac{1 - A_{x:\bar{n}|}}{d} = 5.2$$

The Net Future Loss Random Variable

- **Net future loss (exclude expenses):**

$$L_0^n = \text{PV of benefit outgo} - \text{PV of net premium income}$$

- **Gross future loss (include expenses):**

$$L_0^g = \text{PV of benefit outgo} + \text{PV of expenses} - \text{PV of net premium income}$$

- **Equivalent principle**:

Expected present value of benefits = EPV premiums

(today, all L means L^n)

The Net Future Loss Random Variable

- **Expected value**

$$E[L] = E[Z] - QE[Y]$$

where Z is Insurance r.v. and Y is Annuity r.v.

- **Variance**

$$\begin{aligned} Var[L] &= Var[Z] - Q^2 \cdot Var[Y] - 2Q \cdot COV[Z, Y] \\ &= (1 + \frac{Q}{d})^2 Var[Z] \end{aligned}$$

(for short-term discrete insurance)

The Net Future Loss Random Variable

	Fully-continuous	**Fully-discrete**
Whole life	$(1+\frac{Q}{\delta})^2[{}^2\bar{A}-\bar{A}^2]$	$(1+\frac{Q}{d})^2[{}^2A-A^2]$
endowment insurance (n-yr)	$(1+\frac{Q}{\delta})^2[{}^2\bar{A}_{x:\overline{n}\vert}-\bar{A}^2_{x:\overline{n}\vert}]$	$(1+\frac{Q}{d})^2[{}^2A_{x:\overline{n}\vert}-A^2_{x:\overline{n}\vert}]$

Example

A 3-year fully discrete endowment insurance issued to (x) has death benefit of 1000. Given $q_x = 0.1$, $q_{x+1} = 0.2$, $q_{x+2} = 0.3$, and $i = 0.1$.

- Find the loss random variable L.
- Use the equivalence principle to solve the premium.

Solution: $L =$ PV of benefit outgo $-$ PV of net premium income.

$$L = \begin{cases} 1000v - Q & K_x = 0, \text{prob. } \ q_x = 0.01 \\ 1000v - Q\ddot{a}_{\overline{2}|} & K_x = 1, \text{prob. } {}_{1|}q_x = 0.18 \\ 1000v - Q\ddot{a}_{\overline{3}|} & K_x \geq 2, \text{prob. } {}_2p_x = 0.72 \end{cases}$$

The Net Future Loss Random Variable

Example

A 3-year fully discrete endowment insurance issued to (x) has death benefit of 1000. Given $q_x = 0.1$, $q_{x+1} = 0.2$, $q_{x+2} = 0.3$, and $i = 0.1$.

- Find the loss random variable L.
- Use the equivalence principle premium to solve the premium.

Equivalence Principle $(E[L] = 0)$:

$$0 = 0.01(1000v - Q) + 0.18(1000v - Q\ddot{a}_{\overline{2}|}) + 0.72(1000v - Q\ddot{a}_{\overline{3}|})$$

Compute $\ddot{a}_{\overline{2}|}(\approx 1.9019)$ and $\ddot{a}_{\overline{3}|}(\approx 2.7355)$.

Solve for Q:

$$Q = 323.47$$

The Net Future Loss Random Variable

Example

L is the loss-at-issue random variable for a **fully discrete n-year endowment** insurance of 1 on (x) with premium $P_{x:\bar{n}|}$. Given: (i) ${}^2A_{x:\bar{n}|} = 0.1774$. (ii) $P_{x:\bar{n}|}/d = 0.5850$. Find $Var[L]$.

Solution: directly use the formula

$$Var[L] = [1 + \frac{P_{x:\bar{n}|}}{d}]^2 \cdot [{}^2A_{x:\bar{n}|} - A^2_{x:\bar{n}|}] = 0.103$$

we still need to find $A_{x:\bar{n}|}$: by the equivalence principle again

$$\frac{P_{x:\bar{n}|}}{d} = \frac{A_{x:\bar{n}|}}{1 - A_{x:\bar{n}|}}.$$

(solve $A_{x:\bar{n}|} = 0.3691$)

Percentile Premiums

Example

Consider a fully continuous whole life insurance of 1000 on (x), whose future lifetime T_x has the density

$$f_x(t) = \frac{t}{1250}, \quad 0 \leq t \leq 50.$$

Assume $\delta = 0.05$.

1. If the premium rate is 10 per annum, calculate $E[{}_0L]$ and $P({}_0L > 0)$.
2. What annual premium should the insurer charee so that he will make a profit with 50% probability?

Solution.

$$E({}_0L) = 1000\bar{A}_x - 10\bar{a}_x = 73.678$$

where $\bar{A}_x$ can be computed by

$$\bar{A}_x = E[e^{-0.05T_x}] = \int_0^{50} e^{-0.05}\frac{t}{1250}dt = 0.2280648$$

and

$$\bar{a}_x = \frac{1-\bar{A}_x}{\delta} = 15.438704$$

- $$
\begin{aligned}
P[{}_0L > 0] &= P[e^{-\delta T_x} - \frac{\pi}{S\delta + \pi} > 0] \\
&= P[T_x < -\frac{1}{\delta}\ln(\frac{\pi}{S\delta + \pi})] \\
&= F_x(35.8352) = 0.5137
\end{aligned}
$$

- Make profit $= {}_0L \leq 0$. Find premium rate such that $P({}_0L \leq 0) = 0.5$.

$$
P[T_x < -20\ln(\frac{\pi}{50 + \pi})] = 0.5
$$

$$
400[\ln(\frac{\pi}{50 + \pi})]^2/2500 = 0.5
$$

Solve $\pi = 10.2928$.

Portfolio percentile premium principle: Assume there are N iid future loss random variable $L_{0,i}$. Define

$$L = \sum L_{0,i}.$$

Then it is approximated normal distribution by the central limit theorem. And

$$P[L < 0] = \alpha$$

is easy to compute.

Gross Premium

- **Net future loss (exclude expenses):**

$$L_0^n = \text{PV of benefit outgo} - \text{PV of net premium income}$$

- **Gross future loss (include expenses):**

$$\begin{aligned} L_0^g = \text{PV of benefit outgo} + \text{PV of expenses} \\ - \ \text{PV of net premium income} \end{aligned}$$

- **Equivalent principle**:

$$E[L_0^g] = 0$$

$$\text{EPV of benefits} + \text{EPV of expenses} = \text{EPV gross premiums}$$

Gross Premium

Example (from lecture note, example 1)

A whole life insurance policy for \$1,000 is sold to (65). Pricing basis is the Illustrative Life Table with interest at 6%. Expenses are as follows:

1. Fixed cost of 2 per year (including year 1); plus
2. Variable cost of 6% of gross premium.

Find net premium for the insurance as well as gross premium necessary to cover expenses.

Compute the net premium:

$$\underbrace{\text{EPV of benefits outgo}}_{1000A_{65}} = \underbrace{\text{EPV net premiums income}}_{P\ddot{a}_{65}}$$

$$P = 1000\frac{A_{65}}{\ddot{a}_{65}} \approx 44.44$$

Gross Premium

Example (from lecture note, example 1)

A whole life insurance policy for $1,000 is sold to (65). Pricing basis is the Illustrative Life Table with interest at 6%. Expenses are as follows:

1. Fixed cost of 2 per year (including year 1); plus
2. Variable cost of 6% of gross premium.

Compute the gross premium:

$$\underbrace{\text{EPV of benefits}}_{1000A_{65}} + \underbrace{\text{EPV of expenses}}_{(2+0.06G)\ddot{a}_{65}} = \underbrace{\text{EPV of gross premiums}}_{G\ddot{a}_{65}}$$

$$G = \frac{1000A_{65} + 2\ddot{a}_{65}}{0.94\ddot{a}_{65}} \approx 49.40$$

(Important: compute expenses)

Gross Premium

Example (#9 in Gross premium HW)

For a fully-discrete 5-payment, 10-year deferred, 20-year term insurance of 1000 on (30) you are given the following expenses:

1. Expenses are paid at the beginning of the policy year.
2. Gross premium is determined using the equivalence principle.

Expense type	Year 1		Year 2-10	
	% Premium	Per policy	% Premium	Per policy
Taxes	5	0	5	0
Commission	25	0	10	0
Policy Maintenance	0	20	0	10

Find G, assuming ILT at 6%.

Gross Premium

Expense type	Year 1		Year 2-10	
	% Premium	Per policy	% Premium	Per policy
Taxes	5	0	5	0
Commission	25	0	10	0
Policy Maintenance	0	20	0	10

Find expenses:

$$\text{Expenses on Taxes} = G \cdot 0.05 \cdot a_{30:\overline{5}|}$$
$$\text{Expenses on Commission} = G \cdot 0.15 + G \cdot 0.1 \cdot a_{30:\overline{5}|}$$
$$\text{Exp. on Policy Maint.} = 10\ddot{a}_{30:\overline{10}|} + 10$$

(Need to know the period of each expenses)

Gross Premium

Use Equivalent Principal:

$$\underbrace{\text{EPV of benefits}}_{1000\,{}_{10}E_{30}A^{1}_{40:\overline{20}|}} + \underbrace{\text{EPV of expenses}}_{(\star)} = \underbrace{\text{EPV premiums}}_{G\ddot{a}_{30:\overline{5}|}}$$

$$(\star) = 0.05G \cdot a_{30:\overline{5}|} + 0.15G + 0.1G \cdot a_{30:\overline{5}|} + 10\ddot{a}_{30:\overline{10}|} + 10$$

Example (From lecture note, example 3)

A 3-year policy has the following expenses

	First Year	Renewal
% of Premium	50%	10%
Face amount	\$10 per \$1,000	\$1 per \$1,000
Per policy	\$25	\$5
Settlement Expense	\$10 per policy plus \$1 per \$1,000 face amount	

Find the APV of each expenses.

Settlement Expense (assume face amount is 1000):

$$(10+1)A^{1}_{x:\overline{3}|}$$

Outline

Reserves

t-th Year Terminal Reserve ${}_tV_x$:

$$
\begin{aligned}
{}_tV_x = &\text{ APV future insurance benefits from age (x+t)} \\
&- \text{APV future benefit premiums from age (x+t)}
\end{aligned}
$$

Discrete case:

$$ {}_tV_x = A_{x+t} - P_x \ddot{a}_{x+t} $$

Continuous case:

$$ {}_t\bar{V}_x = \bar{A}_{x+t} - \bar{P}_x \bar{a}_{x+t} $$

Example (# 1 in HW)

Demonstrate the equivalence of the following, all of which are definitions of ${}_tV_x$:

1. $A_{x+t} - P_x \ddot{a}_{x+t}$
2. $1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}$
3. $\frac{P(A_{x+t}) - P(A_x)}{P(A_{x+t}) + d}$
4. $\frac{A_{x+t} - A_x}{1 - A_x}$

Reserves

Important formula:

$$P = \frac{A}{a}$$
$$\ddot{a} = \frac{1-A}{d}$$

- 1) $\iff$ 2)

$$
\begin{aligned}
{}_tV_x &= A_{x+t} - P_x \ddot{a}_{x+t} \\
&= 1 - d\ddot{a}_{x+t} - (\frac{1}{\ddot{a}_x} - d)\ddot{a}_{x+t} \\
&= 1 - \frac{\ddot{a}_{x+t}}{a_t}
\end{aligned}
$$

- 2) $\iff$ 3)

$$1 - \frac{\ddot{a}_{x+t}}{a_t} = 1 - \frac{P(A_x) + d}{P(A_{x+t}) + d} = \frac{P\left(A_{x+t}\right) - P\left(A_x\right)}{P\left(A_{x+t}\right) + d}$$

- 3) $\iff$ 4)

$$\frac{P\left(A_{x+t}\right) - P\left(A_x\right)}{P\left(A_{x+t}\right) + d} = \ldots \text{(on board)}$$

$$= \frac{A_{x+t} - A_x}{1 - A_x}$$

Example

Demonstrate the equivalence of the following, all of which are definitions of ${}_tV_x$:

1. $A_{x+t} - P_x \ddot{a}_{x+t}$
2. $1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}$
3. $\frac{P(A_{x+t}) - P(A_x)}{P(A_{x+t}) + d}$
4. $\frac{A_{x+t} - A_x}{1 - A_x}$

Pg 18. It also works for other types of insurance (n-term, continuous, etc.)

n-year policy value for an h-pay, Whole Life policy issued to (x)

$$^{h}{}_{t}V_{x}$$

Example

Given $P_x = 0.01212$, $^{20}P_x = 0.01508$, $P_{x:\overset{1}{\overline{10|}}} = 0.06942$, and ${}_{10}V_x = 0.11430$. Calculate $^{20}{}_{10}V_x$